(Cubical) Computability Structures

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A type theory is an indexed inductive definition of trees labeled by $\langle \Gamma\ \mathit{ctx} \rangle$, $\langle \Gamma \vdash A\ \mathit{type} \rangle$, $\langle \Gamma \vdash A \equiv B\ \mathit{type} \rangle$, $\langle \Gamma \vdash a : A \rangle$, and $\langle \Gamma \vdash a \equiv b : A \rangle$, satisfying some bizarre conditions.

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First two notions legitimate, but not good definitions of "a type theory"; confusing "has" for "is".

Functorial semantics of type theories

Definition (Uemura [Uem19])

A representable map category is a lex category together with a class of "representable maps" closed under isomorphism, composition, pullback, and along which pushforwards exist.

- Judgmental structure of (non-modal) strict type theories with many judgments == representable map category.
- Hypothetical judgment == pushforward (only available for representable maps!).

Example (Natural model)

Let $\mathbb{T}$ be the smallest type theory containing a representable map $\tilde{\mathbf{T}} \xrightarrow{\tau} \mathbf{T}$. A rep-map functor $\mathbb{T} \longrightarrow \mathcal{P}\mathrm{r}(\mathcal{C})$ is a natural model over $\mathcal{C}$ in the sense of Awodey [Awo18].

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Why do we care about that? Necessary for *implementing* type theoretic languages in a computer (typechecking).

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Example (Injectivity of type constructors)

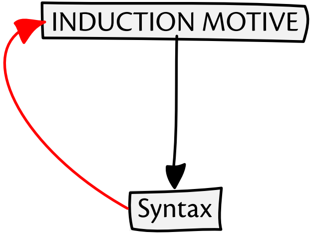
If $\Gamma \vdash A \to B \equiv C \to D\ \mathit{type}$, then $\Gamma \vdash A \equiv C\ \mathit{type}$ and $\Gamma \vdash B \equiv D\ \mathit{type}$.

- Necessary for completeness of algorithmic definitional equality, typechecking.
- Goes beyond the language of representable map categories, therefore not preserved by homomorphisms.

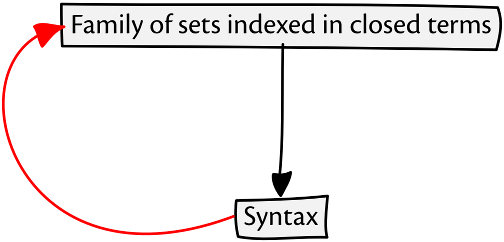
Studying syntax using semantics

First of all, syntax *is* semantics (Lawvere); special because it has a *universal property*.

Interesting theorems about syntax are proved by choosing a family lying over the term model (induction motive), and exhibiting a section using this universal property.



Example: Canonicity [Cro93; Shu15; Coq19]



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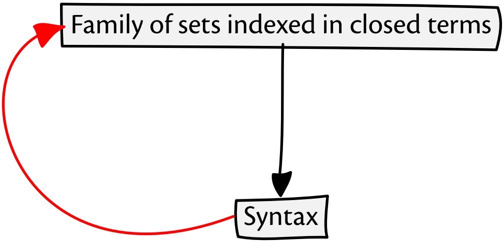

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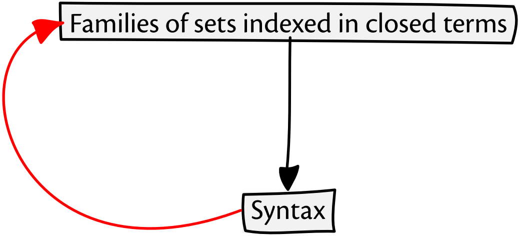
1. Contexts are families of sets indexed in the closed substitutions for a context. For instance, the canonical family $\mathbb{N} \longrightarrow \{\diamond \longrightarrow \mathsf{nat}\}$.
2. Substitutions are morphisms of families of sets, tracked by a syntactic substitution.

The section exhibits for each closed term $\diamond \vdash N : \mathsf{nat}$ a numeral $n \in \mathbb{N}$ encoded by N. □

Example: Cubical Canonicity [Awo15; CHS19; SAG19; SAG20]



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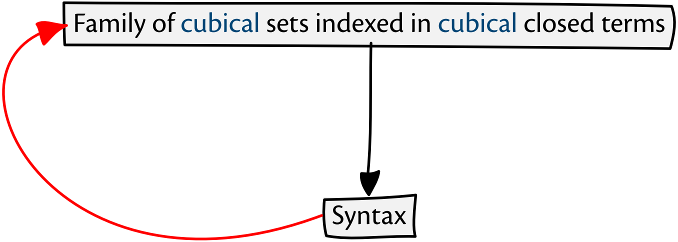
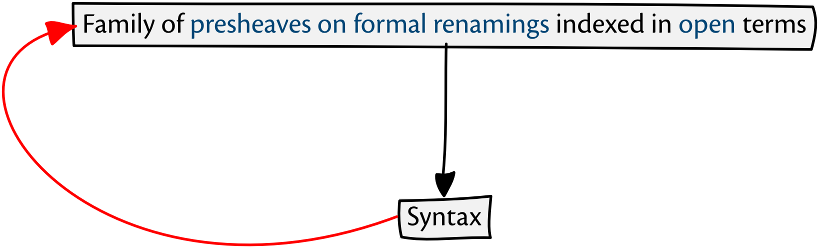


Example: Normalization [AHS95; Fio02; Coq19]



Artin gluing and Tait's method

In the 1960s and 1970s, both geometers and logicians were searching for useful ways to combine existing (spaces, theories, models) into new (spaces, theories, models). Unfortunately they didn't talk to each other much.

- In 1967, Tait develops his ingenious *method of computability* to prove a normalization theorem for typed λ-calculus [Tai67], by attaching *predicates* (proof-irrelevant families) to *raw* terms.
- In the early 1970s, the Grothendieck school develops what is now called *Artin gluing* [AGV72], a way to parameterize a topos in another topos.
- In 1978, Freyd rationalizes Tait's trick as an instance of Artin gluing [Fre78], passing from proof-irrelevant families on raw terms to proof-relevant families on abstract terms.
- In 2015–2020, a flurry of activity in applying gluing to type theory, most notably by Shulman and Coquand [Shu13; Shu15; AK16; SS18; Coq19; KHS19; KS19; CHS19; SAG19; SA20; SAG20].

Artin gluing

Let $\mathcal{C}$ be a structured category of some kind, and let $\mathcal{C} \xrightarrow{F} \mathcal{E}$ be a functor (typically, preserving some finite limits).

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, "\mathrm{gl}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E}^{\to} \arrow[d, "\mathrm{cod}"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{E}
\end{tikzcd}$$

An "Artin gluing theorem" says that $\mathcal{G}$ can be equipped with the same structures, and that $\mathcal{G} \xrightarrow{\mathrm{gl}} \mathcal{C}$ preserves it. We have Artin gluing theorems for many kinds of theory [AGV72; CJ95; KHS19; SA20].

Gluing models of type theory along flat functors [SA20]

Different theorems can be proved by gluing along different functors:

- The global sections functor $\mathcal{C} \longrightarrow \mathbf{Set}$ proves canonicity.
- The Yoneda embedding $\mathcal{C} \longrightarrow \mathcal{P}\mathrm{r}(\mathcal{C})$ proves definability results.
- Let $\mathcal{R} \longrightarrow \mathcal{C}$ be the category of *formal contexts and formal renamings*; the resulting *renaming nerve* functor $\mathcal{C} \longrightarrow \mathcal{P}\mathrm{r}(\mathcal{R})$ proves normalization [Fio02; Coq19].
- Let $\mathcal{C}$ carry an *interval object*, i.e. a f.p. functor $\square \longrightarrow \mathcal{C}$. Resulting *renaming nerve* functor $\mathcal{C} \longrightarrow \mathcal{P}\mathrm{r}(\square)$ proves cubical canonicity [SAG19; SAG20], as suggested by Awodey and Fiore.

General theorem (S., Angiuli): you can glue along *flat functors* from a model of MLTT into a Grothendieck topos [SA20].

Less general than Kaposi, Huber, and Sattler [KHS19], but avoids pernicious/anti-categorical *equality of objects*.

Gluing models of type theory along flat functors [SA20]

Origin of this work was efforts to exploit the classical theory of Artin gluing for Grothendieck topoi, to simplify the corresponding proofs for models of type theory (not usually topoi!!).

Let $\mathcal{C}$ be the category of contexts of a model of MLTT, and let $\mathcal{E}$ be a Grothendieck topos. When does $\mathcal{G}$ carry a model of type theory?

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, "\mathsf{gl}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}^{\to} \arrow[d, "\mathsf{cod}"] \\
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\end{tikzcd}$$

Probably not enough for F to preserve finite limits ($\mathcal{C}$ barely has any limits!). Generally very painful because we must show existence and preservation of type-theoretic structure simultaneously.

Power Move: factor through Yoneda

$$\begin{tikzcd}
\mathcal{G}_K \arrow[r, hook, "K"] \arrow[d, "\mathsf{gl}_K"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G} \arrow[r] \arrow[d, "\mathsf{gl}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}^{\to} \arrow[d, "\mathsf{cod}"] \\
\mathcal{C} \arrow[r, hook, "\mathsf{y}"'] & \mathcal{P}r(\mathcal{C}) \arrow[r, "\hat{F}"'] & \mathcal{E}
\end{tikzcd}$$

- **Artin, Grothendieck, Verdier:** if $\hat{F}$ lex & accessible, then $\mathcal{G}$ a topos and gl a logical morphism [AGV72]!
- **Diaconescu:** $\mathcal{P}r(\mathcal{C})$ is the classifying topos for the theory of flat functors out of $\mathcal{C}$; therefore $\hat{F}$ lex iff F is flat [Bor94].
- **S., Angiuli:** $\mathcal{G}_K \overset{K}{\hookrightarrow} \mathcal{G}$ dense, therefore the nerve $\mathcal{G} \longrightarrow \mathcal{P}r(\mathcal{G}_K)$ fully faithful.

Easier, more abstract! Work in the internal language of $\mathcal{G}$ (“general computability structures), then transfer to model of type theory over $\mathcal{G}_K$ (“compact computability structures”) via nerve.

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Advantage of working in $\mathcal{G}$ atop $\mathcal{P}\mathrm{r}(\mathcal{C})$: the judgmental structure (natural model) of type theory lives in $\mathcal{P}\mathrm{r}(\mathcal{C})$, so we may define a computability structure lying over it directly.

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2. Glue along change of base $\mathcal{P}\mathrm{r}(\mathcal{C}) \xrightarrow{\epsilon^*} \mathcal{P}\mathrm{r}(\mathcal{R})$ to get glued judgments (general computability structures), pull back along Yoneda to get glued contexts (compact computability structures); latter is gluing along cubical nerve $\mathcal{C} \xrightarrow{\mathsf{N}_\epsilon} \mathcal{P}\mathrm{r}(\mathcal{R})$.

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$$
\begin{tikzcd}
\tilde{\mathbf{L}} \arrow[r, "\boldsymbol{\lambda}"] \arrow[d] & \mathbf{L} \arrow[d] & \tilde{\mathbf{R}} \arrow[r, "\boldsymbol{\rho}"] \arrow[d] & \mathbf{R} \arrow[d] & \mathcal{G} \arrow[d] \\
\tilde{\mathbf{T}} \arrow[r, "\boldsymbol{\tau}"'] & \mathbf{T} & \tilde{\mathbf{T}} \arrow[r, "\boldsymbol{\tau}"'] & \mathbf{T} & \mathcal{P}\mathrm{r}(\mathcal{C})
\end{tikzcd}
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Saturation of computability structures

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Require vertical "reify" / "reflect" maps in the language of the gluing fibration:

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\begin{tikzcd}
\mathbf{L} \arrow[r, "\uparrow"] \arrow[dr, mapsto] & \mathcal{G}\langle \mathbf{T} \rangle \arrow[r, "\downarrow"] \arrow[d, mapsto] & \mathbf{R} \arrow[dl, mapsto] & & \mathcal{G} \arrow[d] \\
& \mathbf{T} & & & \mathcal{P}\mathrm{r}(\mathcal{C})
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\uparrow_! \boldsymbol{\lambda} \arrow[r, "\Uparrow"] \arrow[dr, mapsto] & \mathcal{G}\langle \boldsymbol{\tau} \rangle \arrow[r, "\Downarrow"] \arrow[d, mapsto] & \downarrow^* \boldsymbol{\rho} \arrow[dl, mapsto] & \mathcal{G}_{/\mathcal{G}\langle \boldsymbol{\tau} \rangle} \arrow[d] \\
& \boldsymbol{\tau} & & \mathcal{P}\mathrm{r}(\mathcal{C})_{/\boldsymbol{\tau}}
\end{tikzcd}
$$

Applications to programming languages

Programming languages currently understood through transition systems instead of semantics — and it's not for lack of trying!

On the other hand, PL landscape has richer and more sophisticated use of Tait's method/computability than type theory. **Can we attempt reunification?**

Harper & Sterling: Study phase separation in ML module systems using Artin gluing at two levels:

1. Gluing appears to capture the general abstract content of static/dynamic phase separation (after Moggi [Mog89] and Harper, Mitchell, and Moggi [HMM89]).
2. Gluing can be used to prove representation independence / parametricity results for module languages.

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