XTT: Cubical Syntax for Extensional Equality

(without equality reflection)

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- what equations can the machine take responsiblity for? $(\alpha, \delta, \beta, \eta, \xi, \nu, \ldots)$
- what equations induce coercions in terms (silent vs. non-silent)? are they (weakly, strictly) coherent?

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today, we examine XTT: a new take on OTT, using cubes.

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- metatheory: canonicity, decidability of type checking

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- respect

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(many of these can be *defined* in the Agda model of **OTT**, but must be *primitive operations* in “real” **OTT**.)

cubical reconstruction: XTT

goal: find smaller set of primitives which systematically generate (something in the spirit of) **OTT**

idea: start with Cartesian cubical type theory [ABCFHL], restrict to *Bishop sets* à la Coquand [Coq17]

the XTT paper

Sterling, Angiuli, and Gratzer [SAG19]. "Cubical Syntax for Reflection-Free Extensional Equality". *Formal Structures for Computation and Deduction (FSCD 2019).*

see also Chapman, Forsberg, and McBride [CFM18] ("The Box of Delights (Cubical Observational Type Theory)") for the beginnings of a different account of Cubical **OTT**.

(we won't talk about propositions or quotients today. but talk to me about it after! there is a strictness mismatch in both **OTT**,**XTT**.)

XTT: equality using the interval

rather than defining heterogeneous equality by recursion on type structure, define *dependent equality* all at once using a formal interval:

$$\frac{}{0, 1 : \mathbb{I}}$$

EQ FORMATION

$$\frac{i : \mathbb{I} \vdash A : \mathbf{U} \qquad M : A[0] \qquad N : A[1]}{\mathbf{Eq}_{i.A[i]}(M, N) : \mathbf{U}}$$

EQ INTRODUCTION

$$\frac{i : \mathbb{I} \vdash M[i] : A[i] \qquad M[0] = N_0 : A[0] \qquad M[1] = N_1 : A[1]}{\lambda i.M[i] : \mathbf{Eq}_{i.A[i]}(N_0, N_1)}$$

EQ ELIMINATION

$$\frac{M : \mathbf{Eq}_{i.A[i]}(N_0, N_1) \qquad r : \mathbb{I}}{M(r) : A[r] \qquad M(0) = N_0 : A[0] \qquad M(1) = N_1 : A[1]}$$

(along with more β, η rules, etc.)

function extensionality in XTT

we have function extensionality by swapping quantifiers:

$$\frac{F_0, F_1 : A \to B \qquad Q : (x : A) \to \mathbf{Eq}_{\_.B}(F_0(x), F_1(x))}{\lambda i.\lambda x.Q(x)(i) : \mathbf{Eq}_{\_.A\to B}(F_0, F_1)}$$

“respect” is just function application

given $A : \mathbf{U}$ and $x : A \vdash B[x] : \mathbf{U}$ and $Q : \mathbf{Eq}_{\_.A}(M_0, M_1)$, we have:

$$\lambda i.B[Q(i)] : \mathbf{Eq}_{\_.\mathbf{U}}(B[M_0], B[M_1])$$

judgmental UIP via *boundary separation*

in **OTT**, we always have $Q_0 = Q_1 : \mathbf{Eq}(M : A, N : B)$; we achieve this modularly using a *boundary separation*[1] rule:

$$\frac{r : \mathbb{I} \qquad r = 0 \vdash M = N : A \qquad r = 1 \vdash M = N : A}{M = N : A}$$

(does not mention equality type!!)

given $Q_0, Q_1 : \mathbf{Eq}_{i.A}(M, N)$, we have $Q_0 = Q_1 : \mathbf{Eq}_{i.A}(M, N)$ by the β, η, ξ rules of the equality type, together with boundary separation.

[1](it is a presheaf separation condition for a certain coverage on the category of contexts)

generalized coercion: coercion, coherence, and more

we generalize **OTT**'s coercion $[Q] \downarrow_B^A M$ and coherence $[\![Q]\!] \downarrow_B^A M$ with a single operator to coerce between parts of a cube [ABCFHL]:

$$\frac{r, r' : \mathbb{I} \qquad i : \mathbb{I} \vdash A[i] : \mathbf{U} \qquad M : A[r]}{[i.A[i]] \downarrow_{r'}^{r} M : A[r']}$$

given $Q : \mathbf{Eq}_{\_.\mathbf{U}}(A, B)$, we define:

$$[Q] \downarrow_B^A M = [i.Q(i)] \downarrow_1^0 M$$
$$[\![Q]\!] \downarrow_B^A M = \lambda i.[j.Q(j)] \downarrow_i^0 M$$

slogan: coherence is just coercion from a point to a line

like in **OTT** (but unlike **CuTT**), coercion must be calculated by recursion on A, B rather than Q; requires closed universe. ask me why!

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2. ???
3. interpretation into models???

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actually this is totally intractable to do more than once! let's bootstrap it a different way.

objective metatheory and categorical gluing

a new (old) syntax-invariant approach to metatheory

[2] See also Coquand, Huber, and Sattler [CHS19], Kaposi, Huber, and Sattler [KHS19], and Shulman [Shu15].

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the language of category theory makes each of the preceding steps "easy", and independent of syntax / representation details. no raw terms, no PERs.

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cubical gluing: canonicity for XTT

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$$\begin{array}{c} \mathbb{C} \\ \Big\downarrow \mathfrak{U} \\ \square_+ \end{array}$$

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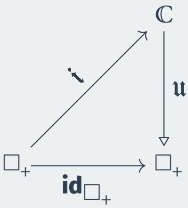

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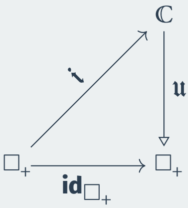

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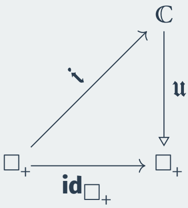

the splitting of $\mathfrak{u}$ interprets dimension substitutions, as well as "relatively terminal" contexts $\mathfrak{i}(\Psi) : \mathbb{C}$ for each $\Psi : \square_+$. we further obtain a "nerve":[3]

$$\mathbf{N} : \mathbb{C} \longrightarrow \mathbf{Pr}(\square_+)$$
$$\mathbf{N}(\Gamma) = \mathbb{C}[\mathfrak{i}(-), \Gamma]$$

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gluing along the cubical nerve

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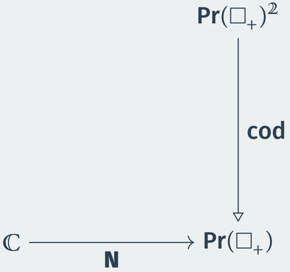

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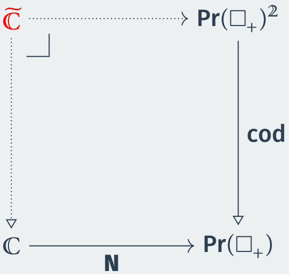

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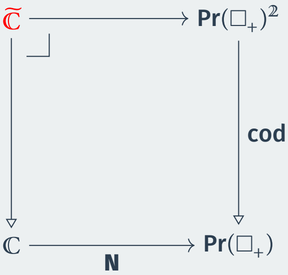

idea: lift the **XTT**-algebra structure from $\mathbb{C}$ to $\widetilde{\mathbb{C}}$, yielding canonicity at base type for *any* representative of the initial **XTT**-algebra $\mathbb{C}$.

summary of contributions

- (Cartesian) cubical reconstruction of **OTT**
- first steps in objective metatheory for cubical type theory
    - algebraic model theory
    - (strict) canonicity by gluing
- next: normalization, decidability of type checking, elaboration!

References I

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