New spaces for denotational semantics

Jonathan Sterling • 2 February 2023 • MSCA Postdoctoral Fellow, Aarhus University

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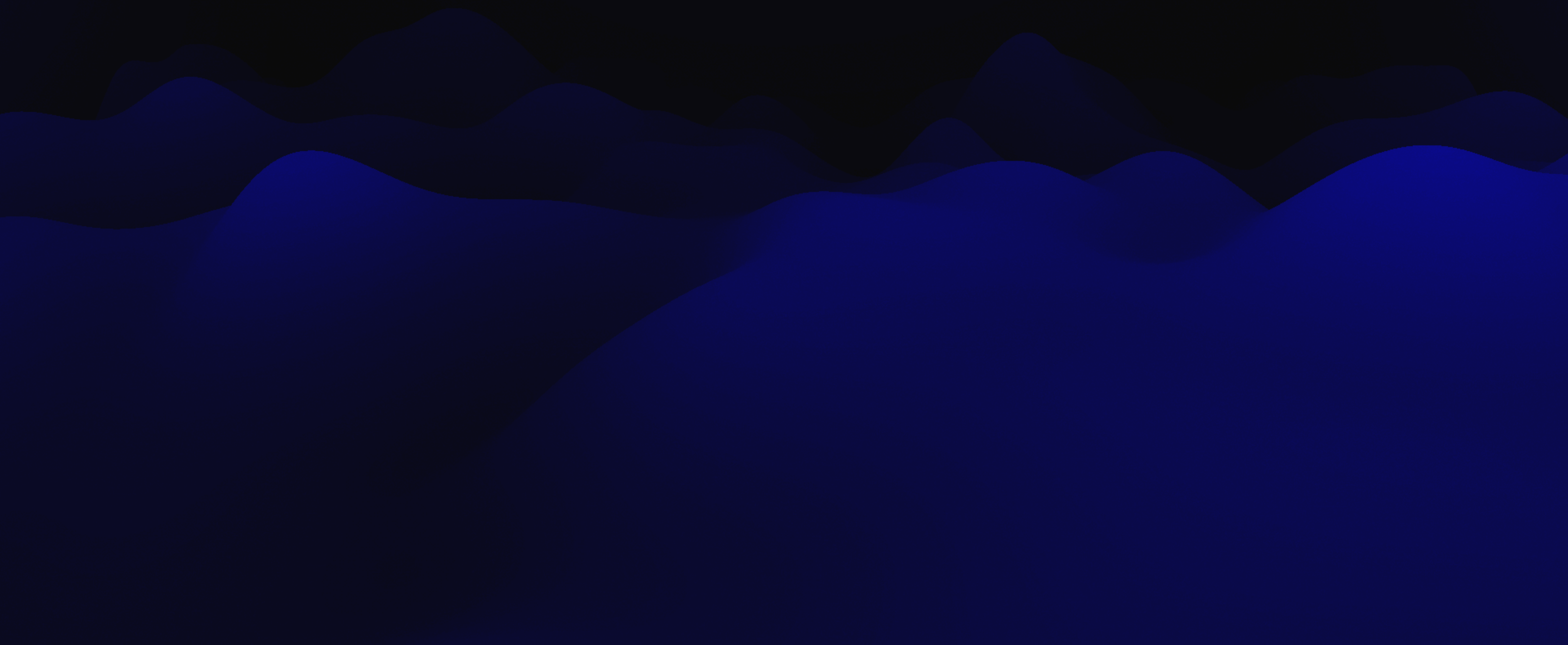

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(Just like physicists create many idealized mathematical models to study different aspects of the material reality of the universe.)

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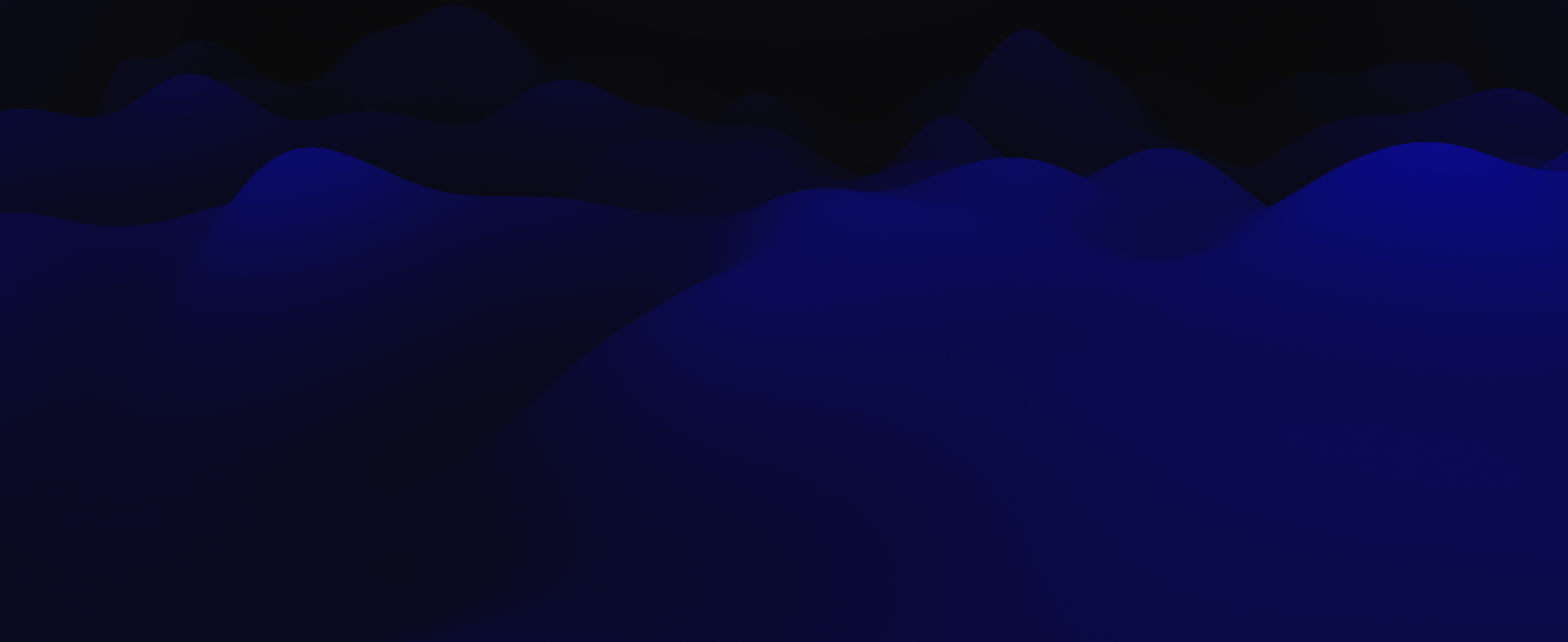

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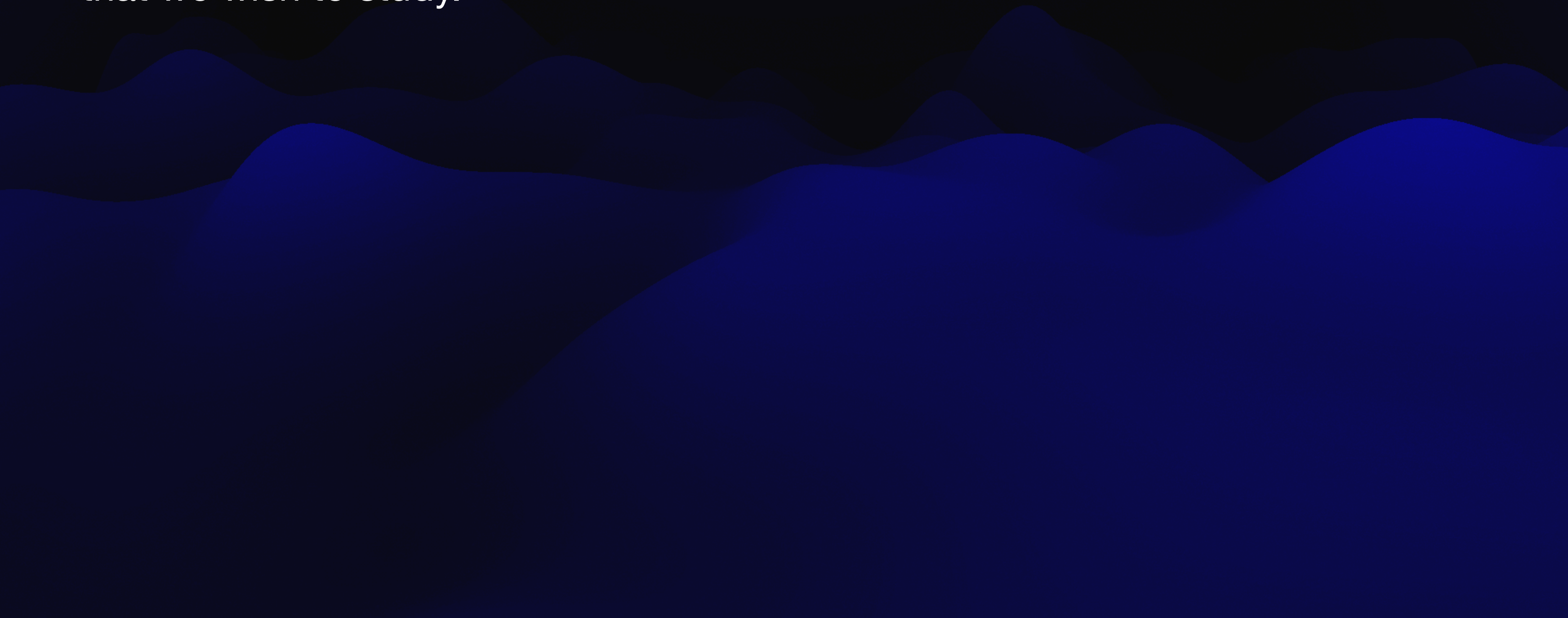
Different models of computation surface different *facets* of program execution that we wish to study.



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Each of these ***mathematical models*** represents a different *abstraction* of the behavior of programs on physical hardware.

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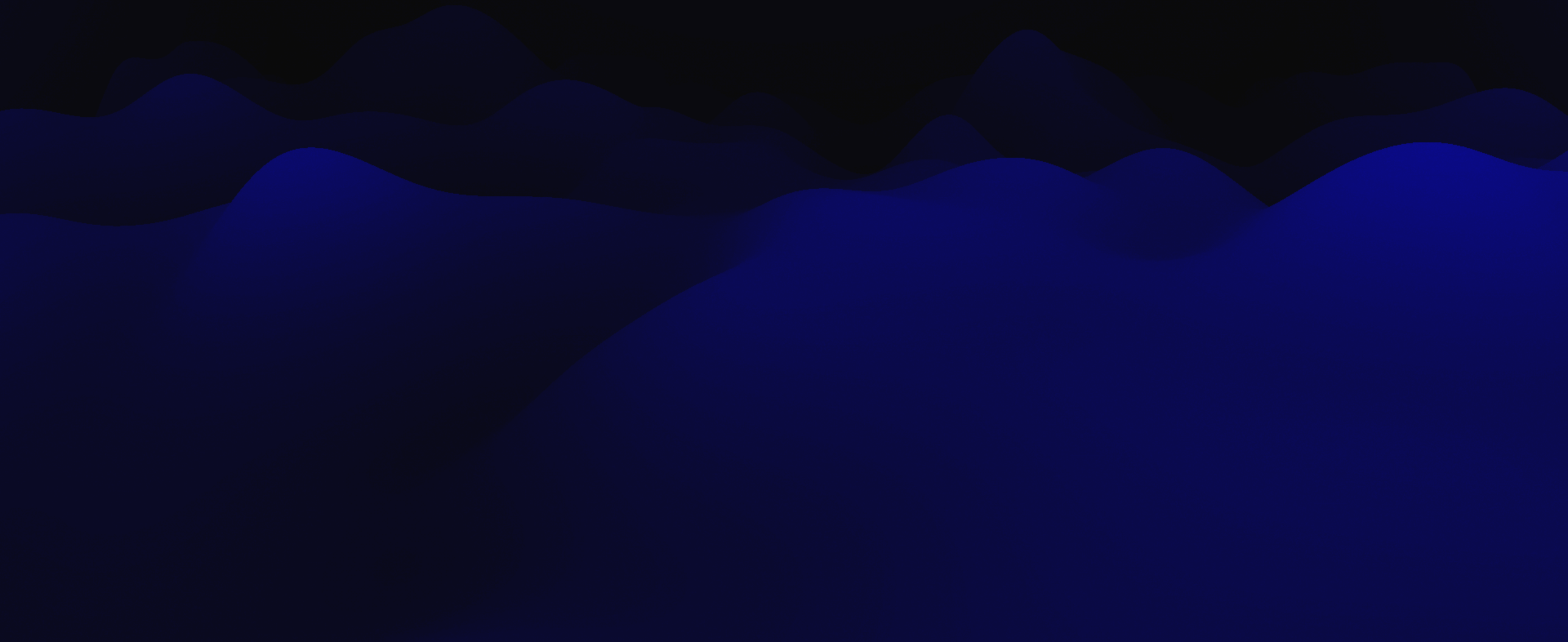

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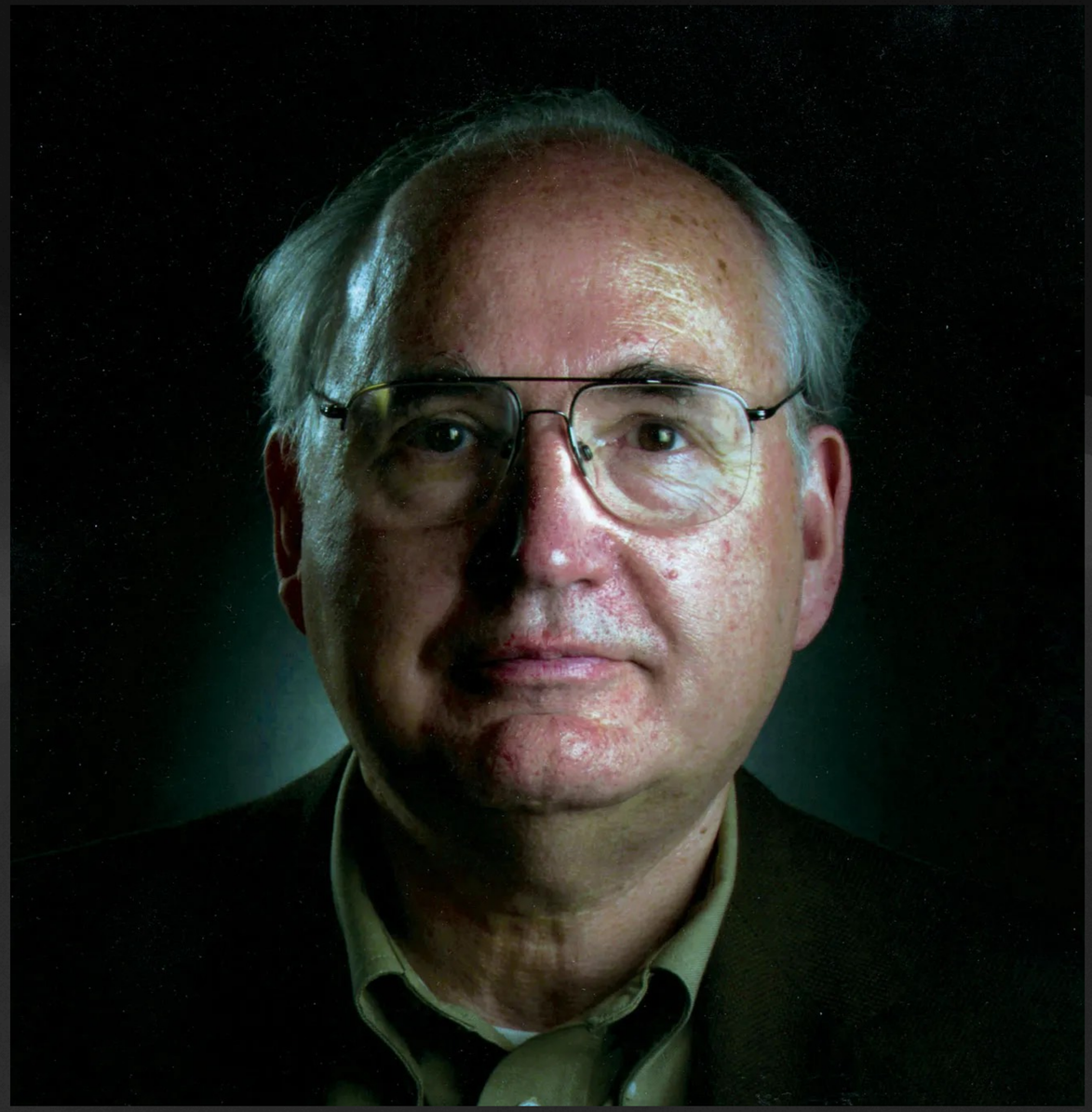
  - This research program has led to the solution of several open problems in dependent type theory.

Technically, these models take place in a **domain theory**, *i.e.* a given variation on the **theory of topological spaces** suitable for studying computation.



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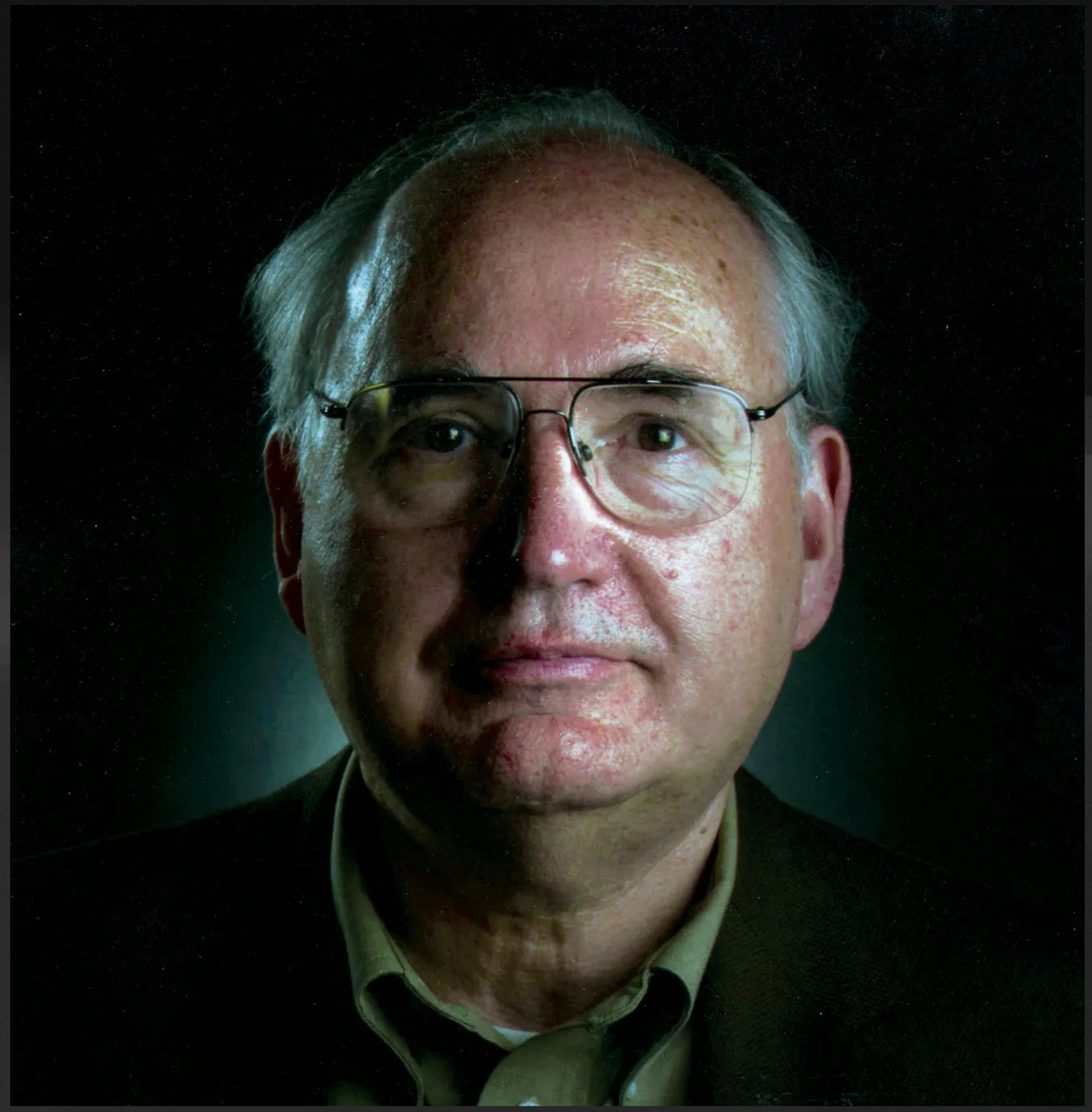

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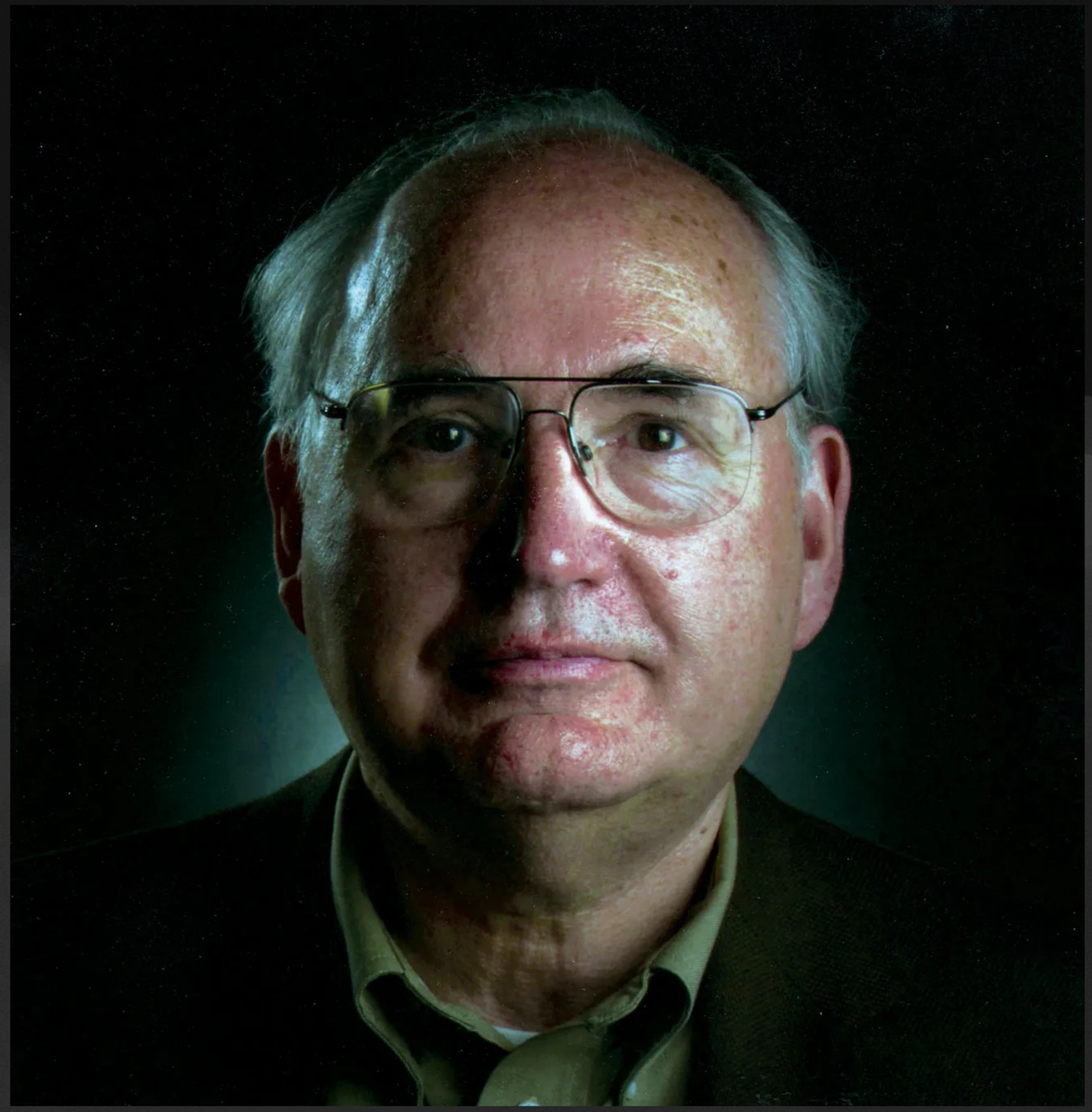
- **Synthetic domain theory:** “sets” in a topos.



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These gluings provide **universal** ways to surface subtle aspects of computation (see also the recent work of Matache, Moss, Staton).

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S. & Harper. *"Sheaf semantics of termination-insensitive noninterference."* **FSCD '22.**

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Niu, S., Grodin, Harper. *"A Cost-Aware Logical Framework."* **POPL '22.**

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- **Solution:** glue together three copies (left, right, and middle) of your computational model. The left and right sides represent two implementations of a data structure, and the “middle” carries a representation invariant between the two.

MIDDLE: Representation Invariant

$$l, r_{front}, r_{back} : \texttt{List} \mid l = \text{append}(r_{front}, \text{reverse}(r_{back}))$$

LEFT: Naïve Queue

$$l : \text{List}$$

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S., Gratzer, Birkedal. *“Denotational semantics of general store and polymorphism.”* Under review.

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Uemura. *"Normalization and coherence for ∞-type theories."* Unpublished manuscript.

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S. *"First Steps in Synthetic Tait Computability: The Objective Metatheory of Cubical Type Theory."*
Ph.D. Thesis, Carnegie Mellon University.

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- ***Denotational semantics of higher-order reference types*** (with Gratzer, Birkedal) in synthetic guarded domain theory; ***higher-order separation logic over denotational semantics*** (with Aagaard, Birkedal).

Thanks!

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