Normalization for (Cartesian) Cubical Type Theory

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   normalization and decidability of judgmental equality for cubical type theory
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   explore the connection between the *phase splitting* in PL and Artin gluing

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Happy to report that we managed to do both.

Sterling, Jonathan and Carlo Angiuli (July 2021). "Normalization for Cubical Type Theory". In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. Los Alamitos, CA, USA: IEEE Computer Society, pp. 1–15. DOI: 10.1109/LICS52264.2021.9470719. arXiv: 2101.11479 [cs.LO].

Sterling, Jonathan and Robert Harper (Oct. 2021). "Logical Relations as Types: Proof-Relevant Parametricity for Program Modules". In: *Journal of the ACM* 68.6. ISSN: 0004-5411. DOI: 10.1145/3474834. arXiv: 2010.08599 [cs.PL].

The cubical hypothesis

HoTT consolidates many *semantical* advances that make type theory more broadly applicable: **univalence, HITs, good quotients, function extensionality, function comprehension!**

But HoTT's equational theory is too weak to compute with. Cubical type theory[1] designed to combine good HoTT semantics with good computational properties.

[1] Bezem, Coquand, and Huber (2014), Angiuli, Hou (Favonia), and Harper (2017), Cohen, Coquand, Huber, and Mörtberg (2017), Awodey (2018), and Angiuli, Brunerie, Coquand, Hou (Favonia), Harper, and Licata (2021)

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Success? We managed to implement it in **redtt** [S., Favonia] and our Swedish colleagues built Cubical Agda. But implementations hinge on **Coquand's conjecture**:

Conjecture (Cohen, Coquand, Huber, and Mörtberg, 2017)

Cubical type theory enjoys normalization and decidable judgmental equality.

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The main ingredient is a new technique called ***synthetic Tait computability*** (STC) abstracting Artin gluing and logical relations.

Computation in □TT: prior art

Prior state of the art (Huber, 2018; Angiuli, Hou (Favonia), and Harper, 2018):

Theorem (Cubical canonicity)

If $\vec{\imath} : \mathbb{I}^n \vdash M(\vec{\imath}) : \mathsf{bool}$ *is a closed n-cube of booleans, then either* $\vec{\imath} : \mathbb{I}^n \vdash M(\vec{\imath}) \equiv \mathsf{tt} : \mathsf{bool}$ *or* $\vec{\imath} : \mathbb{I}^n \vdash M(\vec{\imath}) \equiv \mathsf{ff} : \mathsf{bool}$.

Hence **□TT** is programming language.

Cubical canonicity is only about computation of closed n-cubes. But **implementation** (type checking, elaboration) requires computation in *arbitrary* contexts Γ, *i.e.* normalization.

Main results

I have proved the following suite of results for **□TT** with a countable cumulative hierarchy of universes:[2]

Theorem (Normalization)

There is a computable function assigning to every type $\Gamma \vdash A$ and every term $\Gamma \vdash a : A$ of **□TT** *a unique normal form.*

Corollary (Decidability of equality)

Judgmental equality $\Gamma \vdash A \equiv B$ and $\Gamma \vdash a \equiv b : A$ in **□TT** *is decidable.*

Corollary (Injectivity of type constructors)

If $\Gamma \vdash \Pi(A, B) \equiv \Pi(A', B')$ then $\Gamma \vdash A \equiv A'$ and $\Gamma, x : A \vdash B(x) \equiv B'(x)$.

[2] The preliminary result for **□TT** without universes is j.w.w. Angiuli published in LICS'21 (Sterling and Angiuli, 2021). The full result is in my dissertation (Sterling, 2021).

Proving metatheorems using Tait's method

In 1967, Tait introduced his *method of computability*;[3] Tait computability has remained our only scalable tool for proving metatheorems for logics and type theory (canonicity, normalization, parametricity, *etc.*).[4]

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Idea: an "interpretation" that equips each type A with an predicate $[\![A]\!]$ on elements of A; then show that all *terms* preserve the predicates.

1. First choose the predicate at base type to make soundness of the interpretation imply the desired metatheorem.
2. Then "draw the rest of the owl".

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Operational Tait computability

First define operational semantics $\mapsto^*$ on raw closed terms.

Example (Canonicity)

To prove canonicity, we choose the following predicates:

$$\begin{aligned} [\![\mathsf{bool}]\!](b) &:= (b \mapsto^* \mathsf{tt} \vee b \mapsto^* \mathsf{ff}) \\ [\![A \to B]\!](f) &:= (\forall x : A. [\![A]\!](x) \to [\![B]\!](f(x))) \end{aligned}$$

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(None of the above have satisfactory answers in operational Tait computability.)

The outer limits of operational Tait computability

Specifying and verifying the domain and closure conditions of computability predicates for *cubical* ***canonicity*** proved nearly intractable, *pace* Huber (2018) and Angiuli, Hou (Favonia), and Harper (2018).

Motivated S., Angiuli, and Gratzer to pursue an *algebraic*/gluing-based version of Tait computability for **□TT**[5] à la Coquand (2018), as suggested by Awodey.

Idea: work only with *quotiented* typed terms, make computability predicates proof-relevant. **Outcome:** all difficulties disappeared for cubical canonicity, normalization still required fundamentally new ideas.

Synthetic Tait computability = type theoretic abstraction of the algebraic gluing argument à la Orton and Pitts (2016).

[5] Sterling, Angiuli, and Gratzer (2019)

Introducing synthetic Tait computability

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metatheory	logical relations, Artin gluing	STC

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STC abstracts logical relations by isolating the relationship between syntax and semantics as a pair of modalities.[6]

Expressive enough to recover and simplify existing LR arguments. **More importantly**, STC gave me new geometrical intuitions that I used to solve cubical normalization.

[6] (For experts: STC is the internal language of topoi equipped with open/closed partitions.)

Mixing syntax and semantics

What is really going on in Tait computability? We are *immersing* syntax in a more powerful language (the language of computability predicates) that can express the semantic invariants we want.

(Smoother to develop and use if we generalize to **computability *structures***, *i.e.* **proof-relevant** computability predicates.[7])

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e.g. the computability structure of the booleans:

$$[\![\mathsf{bool}]\!] := \left(x : \boxed{\mathsf{bool}}\right) \times \boxed{\boxed{x = \mathsf{tt}} + \boxed{x = \mathsf{ff}}}$$

[7] *cf.* logical relations for universes and strong sums

Piecing together syntax and semantics

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Computability structures built from syntax and semantics. These can be mixed and matched, but the satisfy some laws:

- Both $\boxed{-}$ and $\boxed{-}$ are *lex idempotent monads.*[8]

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Computability structures built from $\boxed{\text{syntax}}$ and $\boxed{\text{semantics}}$. These can be mixed and matched, but the satisfy some laws:

- Both $\boxed{-}$ and $\boxed{-}$ are *lex idempotent monads.*[8]
- **Complementarity:** semantic things are syntactically trivial, *i.e.* $\boxed{\boxed{A}} \cong$ unit but not the other way around.
- **Fracture:** any computability structure A can be reconstructed from $\boxed{A}$, $\boxed{A}$, and $\boxed{\boxed{A}}$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{A} \arrow[d] \\
\boxed{A} \arrow[r] & \boxed{\boxed{A}}
\end{tikzcd}$$

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The language of synthetic Tait computability

Definition

STC = type theory + modalities $\boxed{-}$/$\boxed{-}$ that behave as above.

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Equivalently, extend type theory by a generic proposition $\P$: **Prop** and define $\boxed{A} := A^{\P}$ and $\boxed{A} := A \cup_{A \times \P} \P$.

Internal language of topoi formed by *Artin gluing* (Artin, Grothendieck, and Verdier, 1972; Wraith, 1974; Rijke, Shulman, and Spitters, 2020).

A recipe for using STC

Analogous to how people use SDG, *etc*. We adapt Kock's recipe:

1. Prove the decisive parts of your theorem synthetically in STC.
2. Choose a topos model of STC (*i.e*. an Artin gluing).
3. Extract your external result from the STC model.

An important part is to choose the right model of STC.

STC models as mapping cylinders

Most useful STC models arise as the *closed mapping cylinder* (Johnstone, 1977) of a morphism of topoi that we think of as a “figure shape” $\boldsymbol{\alpha} : \mathbf{A} \to \widehat{\mathcal{T}}$:[9]

$$\begin{tikzcd}
& \mathbf{A} \arrow[r, "\boldsymbol{\alpha}"] \arrow[d, hook, "\textit{(open immersion)}"'] & \widehat{\mathcal{T}} \arrow[d, hook, dotted, "\textit{(open immersion)}"] \\
\mathbf{A} \arrow[r, hook] \arrow[rr, hook, bend right, "\textit{(closed immersion)}"'] & \mathbf{A} \times \mathbb{S} \arrow[r, dotted] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbf{M}_{\boldsymbol{\alpha}}^{\bullet}
\end{tikzcd}$$

Above $\widehat{\mathcal{T}}$ is the “syntactic topos”. What do we mean by “figure shape”, and how do we choose it?

[9] Equivalently, this is the Artin gluing $\{\mathbf{Set}_{\mathbf{A}}\} \downarrow \boldsymbol{\alpha}^*$ of the inverse image functor $\boldsymbol{\alpha}^* : \mathbf{Set}_{\widehat{\mathcal{T}}} \to \mathbf{Set}_{\mathbf{A}}$.

Choosing a figure shape, abstractly

Let's say we are proving something about the sort $\mathbf{Tp} : \mathcal{T}$ of types. Usually we cannot state or prove our theorem for *all* figures $X \to \mathbf{Tp}$ but only for certain figures, *e.g.* only point-shaped figures (canonicity) or context-shaped figures (normalization).

A figure shape $\boldsymbol{\alpha} : \mathbf{A} \to \widehat{\mathcal{T}}$ is chosen to restrict syntactic objects like $\mathbf{Tp}$ to their "functors of $\mathbf{A}$-shaped points" where $\mathbf{A}$ embodies the permitted figures.

$$\begin{tikzcd}
\boldsymbol{\alpha}^* \mathrm{y}(\mathbf{Tp}) \arrow[r] \arrow[d, maps to] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{y}(\mathbf{Tp}) \arrow[d, maps to] \\
\mathbf{A} \arrow[r, "\boldsymbol{\alpha}"'] & \widehat{\mathcal{T}}
\end{tikzcd}$$

[This is what was going on in the 1990s literature, "Kripke relations of varying arity" (Jung and Tiuryn, 1993; Fiore, 2002).]

Choosing a figure shape, concretely

$$\Psi \Vdash M : A \Downarrow V$$

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element

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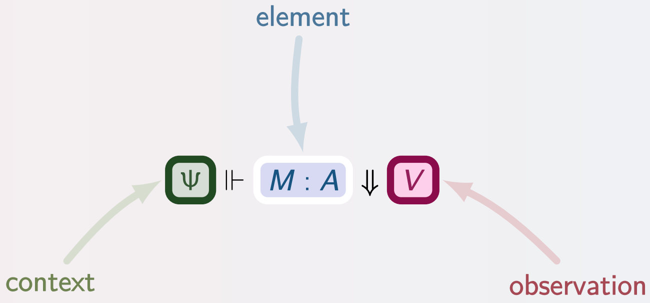
$$\boxed{\Psi} \Vdash \boxed{M : A} \Downarrow \boxed{V}$$

observation

Choosing a figure shape, concretely



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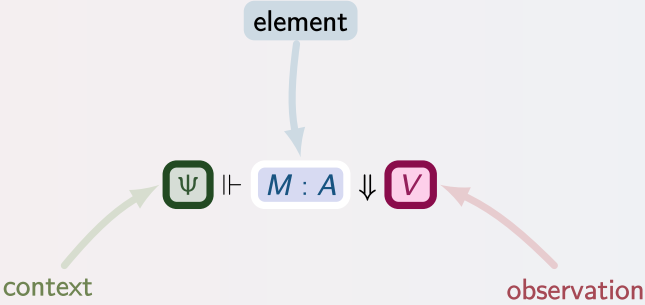

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canonicity: $A \in \{\mathsf{nat}\}$; **normalization:** $A \in \{\Psi \vdash \mathit{type}\}$

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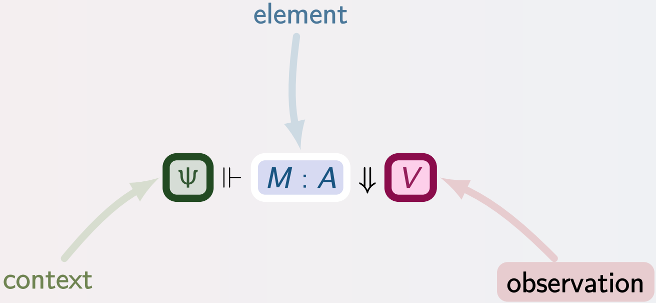

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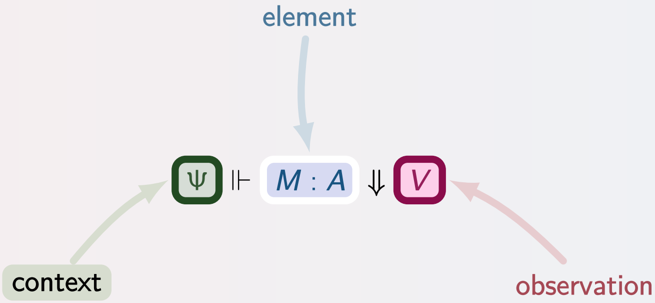
canonicity: $V \in \mathbb{N}$; **normalization:** $V \in \left\{\Psi \Vdash^{\beta\eta}_{\mathbf{nf}} A\right\}$

Choosing a figure shape, concretely



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canonicity: $\Gamma \in \{\cdot\}$; **cubical canonicity:** $\Gamma \in \{\mathbb{I}^n \mid n \in \mathbb{N}\}$; **normalization:** $\Gamma \in \{\vdash \mathit{ctx}\}$

Stability (or lack thereof) of observation

$$\boxed{x : \mathsf{nat}} \Vdash \boxed{x : \mathsf{nat}} \Downarrow \boxed{\mathbf{var}(x)}$$

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$$\boxed{\cdot} \xrightarrow{[\mathsf{fib}(9)/x]} \boxed{x : \mathsf{nat}} \Vdash \boxed{x : \mathsf{nat}} \Downarrow \boxed{\mathbf{var}(x)}$$

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$$\begin{tikzcd} \boxed{\cdot} \arrow[r, "{[\mathsf{fib}(9)/x]}"] & \boxed{x : \mathsf{nat}} & \Vdash & \boxed{x : \mathsf{nat}} \arrow[d, rightsquigarrow] & \Downarrow & \boxed{\mathbf{var}(x)} \\ & & & \boxed{\mathsf{fib}(9) : \mathsf{nat}} & & \end{tikzcd}$$

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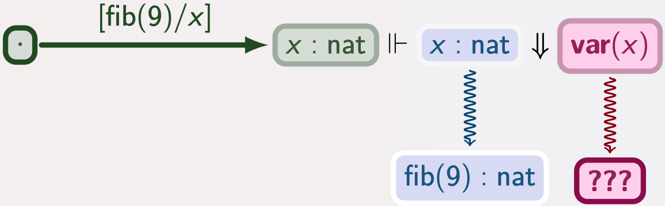

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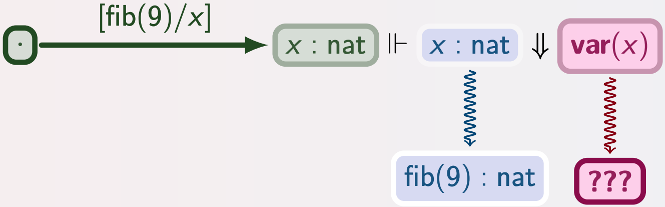

In plain type theory, neutral observations (elimination forms blocked on variables) are closed under *renaming*, but not full substitution.

Therefore normalization takes place over the category $\mathcal{R}$ of contexts and *structural renamings* (weakening, swapping, contraction).

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\end{tikzcd}$$

We shouldn't remove $[0/i], [1/i]$ from the category of contexts and renamings because we need $\mathbb{I}$ to restrict to something *representable* in $\mathrm{Pr}(\mathcal{R})$, *c.f.* **tininess** criterion (Licata, Orton, Pitts, and Spitters, 2018).

The power of dialectical thinking: geometrical negation

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Therefore we define an inductive family $\mathbf{Ne}_{\phi}(A)$ with $\boxed{\mathbf{Ne}_{\phi}(A)} \cong \boxed{A}$ comprised of neutrals e with $\partial e = \phi$. Traditional neutrals $\mathbf{Ne}_{\bot}(A)$; to model destabilization, $\mathbf{Ne}_{\top}(A) \cong \boxed{A}$.

Normalization via Tait's yoga

Tait (1967) introduced the famous *saturation yoga* for normalization:

$$\mathbf{Ne}(A) \subseteq [\![A]\!] \subseteq \mathbf{Nf}(A)$$

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\mathbf{Ne}(A) \arrow[r, "\uparrow_A"] \arrow[dr] & \llbracket A \rrbracket \arrow[r, "\downarrow_A"] \arrow[d] & \mathbf{Nf}(A) \arrow[dl] \\
 & A &
\end{tikzcd}$$

[10] *cf.* normalization by evaluation in the style of Fiore (2002), Altenkirch, Hofmann, and Streicher (1995), Altenkirch and Kaposi (2016), and Coquand (2019)

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"reflection"

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“saturation / Tait’s yoga”

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Yogic injury: unstable neutrals

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 & A &
\end{tikzcd}$$

What if $\phi = \top$?

Yogic injury: unstable neutrals

$$\begin{tikzcd}
\mathbf{Ne}_{\top}(A) \arrow[r, "\uparrow_A"] \arrow[dr] & \llbracket A \rrbracket \arrow[r, "\downarrow_A"] \arrow[d] & \mathbf{Nf}(A) \arrow[dl] \\
& A &
\end{tikzcd}$$

What if $\phi = \top$?

Yogic injury: unstable neutrals

$$\begin{tikzcd}
A \arrow[r, "???"] \arrow[dr, dash] & \llbracket A \rrbracket \arrow[r, "\downarrow_A"] \arrow[d] & \mathbf{Nf}(A) \arrow[dl] \\
& A &
\end{tikzcd}$$

What if $\phi = \top$?

Yogic injury: unstable neutrals

$$\begin{tikzcd}
A \arrow[r, "???"] \arrow[dr, no head] & {[\![A]\!]} \arrow[r, "\downarrow_A"] \arrow[d] & \mathbf{Nf}(A) \arrow[dl] \\
 & A &
\end{tikzcd}$$

What if $\phi = \top$? We must strengthen the “induction hypothesis”.

Stabilization of neutrals

To strengthen the Tait reflection hypothesis, we **glue** unstable neutrals together with compatible computability data along their frontiers of instability.

$$\begin{tikzcd}
\mathbf{Ne}_\phi(A) \wr_\phi \llbracket A \rrbracket \arrow[r, dotted] \arrow[dd, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Ne}_\phi(A) \arrow[d] \\
& A \arrow[d] \\
\phi \Rightarrow \llbracket A \rrbracket \arrow[r] & \phi \Rightarrow A
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\end{tikzcd}$$

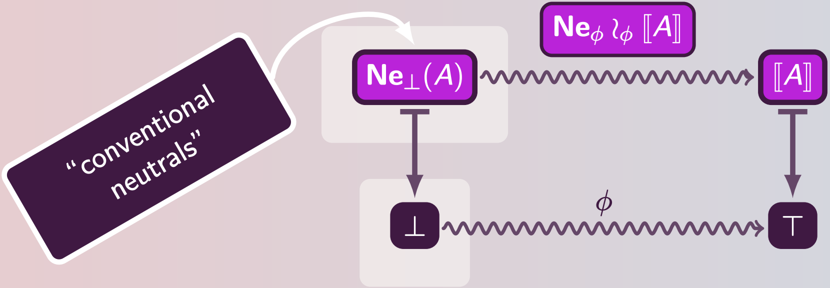
“stabilized neutrals”

A spectrum of computability data

$$\begin{tikzcd}
\mathbf{Ne}_{\bot}(A) \arrow[r, rightsquigarrow, "\mathbf{Ne}_{\phi} \wr_{\phi} \llbracket A \rrbracket"] \arrow[d, mapsto] & \llbracket A \rrbracket \arrow[d, mapsto] \\
\bot \arrow[r, rightsquigarrow, "\phi"] & \top
\end{tikzcd}$$

Stabilization **interpolates** between neutrals and computability data.

A spectrum of computability data



$$\begin{tikzcd}
\mathbf{Ne}_{\perp}(A) \arrow[rr, rightsquigarrow, "\mathbf{Ne}_{\phi} \wr_{\phi} \llbracket A \rrbracket"] \arrow[d, mapsto] & & \llbracket A \rrbracket \arrow[d, mapsto] \\
\perp \arrow[rr, rightsquigarrow, "\phi"] & & \top
\end{tikzcd}$$

“conventional neutrals”

Stabilization **interpolates** between neutrals and computability data.

A spectrum of computability data

“conventional neutrals”

$$\begin{tikzcd}
\mathbf{Ne}_{\perp}(A) \arrow[r, squiggly, "\mathbf{Ne}_{\phi} \wr_{\phi} \llbracket A \rrbracket"] \arrow[d, mapsto] & \llbracket A \rrbracket \arrow[d, mapsto] \\
\perp \arrow[r, squiggly, "\phi"] & \top
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“computability data”

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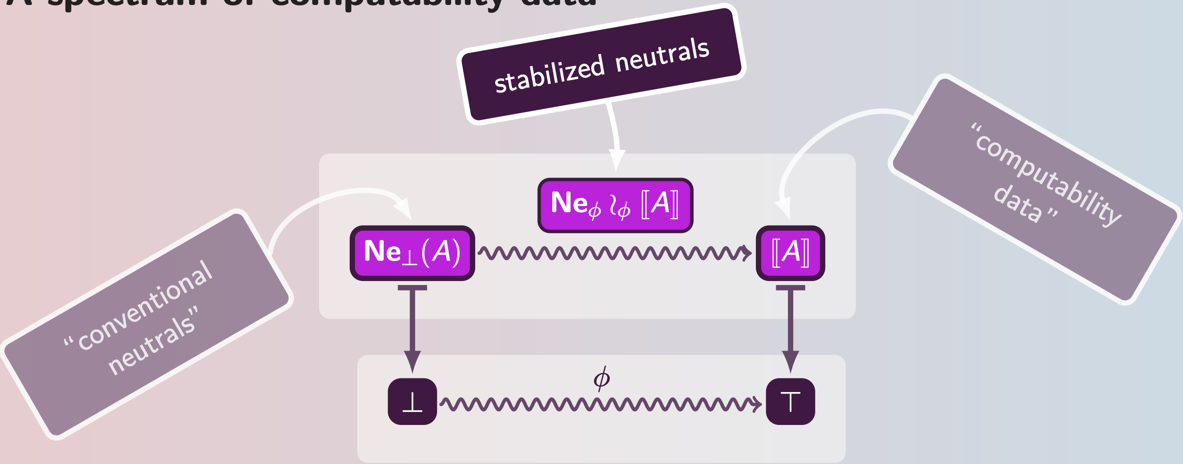

stabilized neutrals

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$\mathbf{Ne}_{\phi} \wr_{\phi} [\![A]\!]$

$$\begin{tikzcd}
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The stabilized Tait yoga

$$\begin{tikzcd}
 & \phi \Rightarrow \llbracket A \rrbracket & \\
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\mathbf{Ne}_{\phi}(A) \arrow[r] & A &
\end{tikzcd}$$

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Lemma (Saturation)

Every type of **□TT** *is closed under the **stabilized** Tait yoga.*

Summary of results

Lemma (Saturation)

Every type of **□TT** *is closed under the* ***stabilized*** *Tait yoga.*

The above is employed to obtain our main results:

Theorem (Normalization)

There is a computable function assigning to every type $\Gamma \vdash A$ *and every term* $\Gamma \vdash a : A$ *of* **□TT** *a unique normal form.*

Corollary (Decidability of equality)

Judgmental equality $\Gamma \vdash A \equiv B$ *and* $\Gamma \vdash a \equiv b : A$ *in* **□TT** *is decidable.*

Corollary (Injectivity of type constructors)

If $\Gamma \vdash \Pi(A, B) \equiv \Pi(A', B')$ *then* $\Gamma \vdash A \equiv A'$ *and* $\Gamma, x : A \vdash B(x) \equiv B'(x)$*.*

A computational conspectus on cubes. . .

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4. **computational interpretation of open terms**
   by Sterling and Angiuli (2021) and Sterling (2021).

What's next for cubical type theory?

We have done more than enough cubical type *theory*. Time for applications!

- **applications to programming and verification**
  Cavallo and Harper (2020), Angiuli, Cavallo, Mörtberg, and Zeuner (2021), and Kidney and Wu (2021)
- **applications to denotational semantics**
  Møgelberg and Veltri (2019), Veltri and Vezzosi (2020), Møgelberg and Vezzosi (2021), and Diezel and Goncharov (2020)
- **applications to ordinary mathematics**
  Forsberg, Xu, and Ghani (2020)
- **applications to synthetic homotopy theory**
  Mörtberg and Pujet (2020), Cavallo (2021), and Brunerie, Ljungström, and Mörtberg (2021)

The era of synthetic Tait computability

- [POPL'22] **A cost-aware logical framework** (Niu, Sterling, Grodin, and Harper)
- [LICS'21] **Normalization for cubical type theory** (Sterling and Angiuli)
- [J.ACM] **Logical Relations As Types: Proof-Relevant Parametricity for Program Modules** (Sterling and Harper)
- **Normalization for multi-modal type theory** (Gratzer)

Let a hundred phase distinctions bloom!

STC also leads to new perspectives on classic PL problems, *cf.* S. and Harper's analysis of the static/dynamic **phase distinction** and sealing in terms of STC.

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logical relations	syntax	semantics
program modules	static	dynamic
security / IFC	public	classified
type refinements	computation	specification
resource analysis	behavior	complexity

Sterling, Jonathan and Robert Harper (Oct. 2021). "Logical Relations as Types: Proof-Relevant Parametricity for Program Modules". In: *Journal of the ACM* 68.6. ISSN: 0004-5411. DOI: 10.1145/3474834. arXiv: 2010.08599 [cs.PL].

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Gratzer, Daniel (2021). *Normalization for Multimodal Type Theory*. arXiv: 2106.01414 [cs.LO].

Sterling, Jonathan and Carlo Angiuli (July 2021). "Normalization for Cubical Type Theory". In: *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. Los Alamitos, CA, USA: IEEE Computer Society, pp. 1–15. DOI: 10.1109/LICS52264.2021.9470719. arXiv: 2101.11479 [cs.LO].

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Niu, Yue, Jonathan Sterling, Harrison Grodin, and Robert Harper (2021). *A cost-aware logical framework*. Conditionally accepted to POPL '22. arXiv: `2107.04663 [cs.PL]`.

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Sterling, Jonathan, Stephanie Balzer, and Robert Harper (2021). "Abstract phase distinctions and noninterference". Work in progress.

Thanks!

“What about Brunerie’s number?”

I was hoping someone would ask that. (-:

1. **It would be great to compute it!** More “compute power” is not the answer, better algorithms and optimizations needed.
2. **It is unrelated to the normalization result**, because normalization is not optimized for computation of closed terms. An evaluator that can efficiently compute Brunerie’s number is not well-adapted for normalization, and vice versa.
3. **Brunerie’s number is not a good benchmark**, exactly analogous to “one plus the Collatz function applied to the one hundred trillionth Fibonacci number” — both probably compute to 2, but no surprise that this takes a lot of time & space.
4. **Whoever computes it will get an feature article in *Quanta***, but the result will not change the landscape for computational applications of cubical type theory.

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