classifying topoi in synthetic guarded domain theory:

the universal property of multiclock guarded recursion

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SYNTHETIC GUARDED DOMAIN THEORY

a new kind of domain theory,
taking place in the internal language of a topos $\mathcal{E}$,
based on a <u>modality</u> for stratification or approximation:

"later modality"

$\blacktriangleright : \mathcal{E}_{/-} \to \mathcal{E}_{/-}$

$\mathsf{next} : 1 \to \blacktriangleright$

$\mathcal{E} \models \forall f : \blacktriangleright X \Rightarrow X.$
$\exists! \bar{f} : X.$
$f(\mathsf{next}_X \bar{f}) = \bar{f}$

("guarded fixed point property")

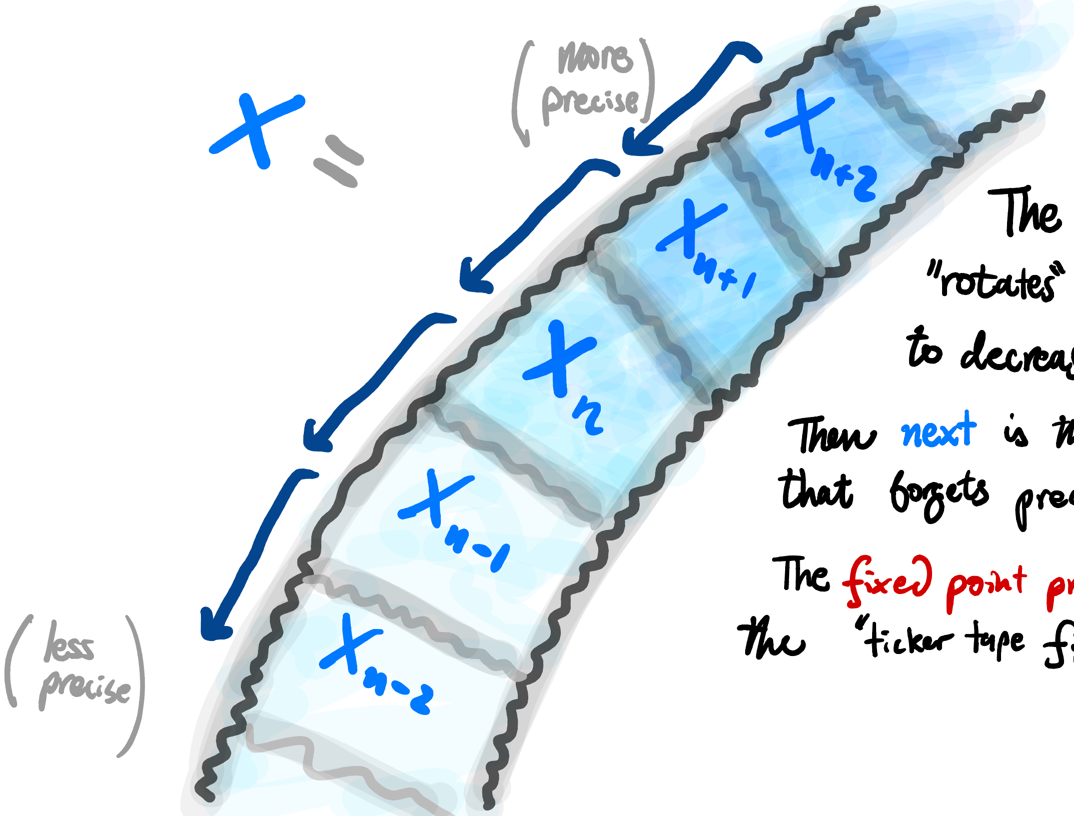
/Löb induction

...

$\blacktriangleright$ is a dependent applicative functor.

APPROXIMATION IN SGDT

Every object $X \in \mathcal{E}$ is a "ticker tape" of approximations:



The later modality $\blacktriangleright$ "rotates" this ticker tape to decrease precision.

Then next is the transition map that forgets precision.

The fixed point principle justified when the "ticker tape figure shape" is well-founded.

GUARDED RECURSIVE TYPES

Recall that recursive types are fixed points $X \cong FX$ that simultaneously carry the structure of the initial algebra $FX \xrightarrow{\sim} X$ and the terminal coalgebra $X \xrightarrow{\sim} FX$. Also called <u>free algebras</u>.

Let $\mathcal{U} \in \mathcal{E}$ be a <u>universe</u>. The f.p. principle implies free algebras for any $F : \mathcal{U} \to \mathcal{U}$ such that

$$\begin{tikzcd} \mathcal{U} \arrow[r, "F"] \arrow[d, "\mathrm{next}_{\mathcal{U}}"'] & \mathcal{U} \\ \blacktriangleright\mathcal{U} \arrow[ur, dashed, "\exists F'"'] & \end{tikzcd}$$

<u>EXAMPLE.</u> (GUARDED STREAMS) Choose $F_{\mathcal{S}_g} X = 2 \times \blacktriangleright X$; then we have a free algebra $2 \times \blacktriangleright \mathcal{S}_g \xrightarrow{\sim} \mathcal{S}_g$. TAIL FUNCTION MUST LOSE PRECISION:

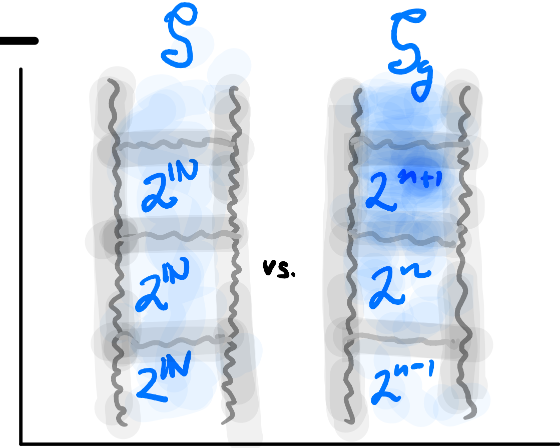
$$\mathrm{tail}_g : \mathcal{S}_g \to \blacktriangleright \mathcal{S}_g.$$

GUARDED STREAMS vs. STREAMS

Guarded streams are not the same as ordinary streams:

$$F_{str} X = 2 \times X \qquad \mathcal{S} \xrightarrow{\sim} 2 \times \mathcal{S}$$

final (not free!) **coalgebra**



With **streams** we have projections $nth : \mathcal{S} \Rightarrow \mathbb{N} \Rightarrow 2$; with **guarded streams** we have only **approximate** projections $nth_g : \mathcal{S}_g \Rightarrow \prod_{n:\mathbb{N}} \blacktriangleright^n 2$.

Guarded streams advantageous because **total** functions $\mathcal{S}_g \to \mathcal{S}_g$ are easily programmed using **f.p. principle** without brittle syntactic checks.

How to adapt to (unguarded) stream programming?

MULTICLOCK GUARDED RECURSION

Atkey & McBride proposed parameterizing the later modality in an abstract collection of "clocks" $\mathbb{K}$:

$$\blacktriangleright : \mathcal{E}/_{\mathbb{K}\times -} \longrightarrow \mathcal{E}/_{\mathbb{K}\times -}$$

Then, taking a dependent product over $\mathbb{K}$ "removes" the later modality:

$$\left(\prod_{k:\mathbb{K}} \blacktriangleright_k A\right) \cong \left(\prod_{k:\mathbb{K}} A\right)$$

So if $\mathbb{K}$ is connected, we may recover stream types from guarded stream types:

$$\mathcal{S}_g^k \cong 2 \times \blacktriangleright_k \mathcal{S}_g^k \qquad \left(\prod_{k:\mathbb{K}} \mathcal{S}_g^k\right) \cong \mathcal{S}$$

TOPOS MODELS OF CLOCKS

AEM defined a programming language rather than a categorical semantics.

Subsequently, Bizjak & Møgelberg and Sterling & Harper developed satisfactory (but very complex/concrete) models of clocks in presheaf topoi.

OUR CONTRIBUTION:

a universal property for the process that adds clocks to a topos model of SGDT in the bicategory of bounded topoi and geometric morphisms over an arbitrary base elementary topos $\mathcal{B}$ w/ n.n.o.

Theorem. The Bizjak–Møgelberg topos $\mathbf{BM}$ is the partial product $\Pi_p\,\hat{\omega}$ of the universal étale geometric morphism $\begin{array}{c}\mathbb{A}_\bullet \\ \downarrow p \\ \mathbb{A}\end{array}$ with the topos of trees $\hat{\omega}$, i.e. the lower bagtopos $\mathbf{BM} = \mathbf{Bag}(\hat{\omega})$.

Vickers; Johnstone

$\blacksquare$

Corollary. A geometric morphism $X \xrightarrow{f} \mathbf{BM}$ is given by exactly

1) an object $K \in X$, and
2) a K-indexed filter on the poset ω.

$\blacksquare$

MAIN RESULT:

Let $\mathcal{B}$ be an elementary topos w/ n.n.o and let $\mathcal{E}$ be a bounded topos over $\mathcal{B}$ carrying the structure of a model of SGDT. Then $\mathrm{Bag}(\mathcal{E}) = P_p \mathcal{E}$ is a bounded topos model of multiclock SGDT over $\mathcal{B}$. $\blacksquare$

Other results:

* useful constructive generalization of the results of Birkedal et al. (First Steps in SGDT) to the relative bounded topos theory over any base topos w/ n.n.o;
* closure of SGDT models under internal presheaves and left exact localization.