Objective Metatheory of (Cubical) Type Theory

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August 31, 2020

The implementation and semantics of dependent type theories can be studied in a syntax-independent way. *Using the semantic techniques of the* ***objective metatheory****, type theorists can obtain succinct and conceptual proofs of formerly intractable results.*

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About the syntax of type theory. *But, by soundness, also about any model of type theory.*

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$$\frac{\Gamma \vdash A \equiv A' \ \ \mathit{type} \qquad \Gamma \vdash B \equiv B' \ \ \mathit{type}}{\Gamma \vdash (A \to B) \equiv (A' \to B') \ \ \mathit{type}}$$

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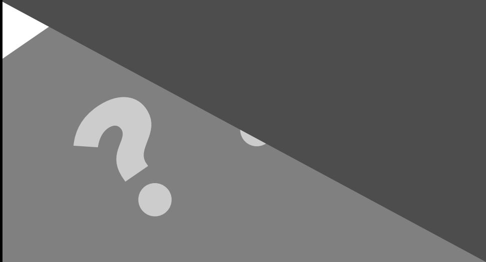

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Why do we care?

Admissibilities like this are why it is even possible to implement type checkers!

Most metatheorems important for implementation are consequences of *normalization*.

Even for basic type theory, normalization is hard to prove rigorously: 100-200 pages(*) of single-use technical lemmas that seem to have nothing to do with the matter at hand.

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All these are solved by cubical type theory (*).

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(*) Angiuli, Hou (Favonia), and Harper [AHH17], Angiuli, Brunerie, Coquand, Hou (Favonia), Harper, and Licata [Ang+17], Awodey [Awo18a], Cohen, Coquand, Huber, and Mörtberg [Coh+17], Huber [Hub18], and Orton and Pitts [OP16], …

Implementations of cubical type theory!

Several variants of cubical type theory have been implemented.

Gen.	Style	Implementation
0	evaluator	`cubical`
1	typechecker PRL	`cubicaltt`, `yacctt` **RedPRL**
2	proof assistant	Cubical Agda, `redtt`, `cooltt`

[Ang+18b; Coh+15; Coh+18; MA18; Red18; Red20; VMA19]

Ours: **RedPRL**, `redtt`, `cooltt`.

Our implementations: RedPRL, redtt, cooltt

Each implementation was tied to a scientific experiment!

	Premise	Result
RedPRL	The PRL methodology benefits HTT implementation.	PRL impedes HTT implementation.
redtt	Interactive cubical refinement + decidable(?) jdg.eq. increases usability.	Confirmed.
cooltt	LF formulation + more extensional equality on $\mathbb{F}$ & systems suitable for efficient implementation.	Early positive indications.

My contributions: generalization of interactive proof with holes to account for cubical boundaries; more efficient algorithms for cubical evaluation.

What is cubical type theory?

An extension of Martin-Löf type theory!

1. Interval object $\mathbb{I}$, classifying "dimensions":

$$\frac{}{0 : \mathbb{I}} \qquad \frac{}{1 : \mathbb{I}} \qquad \frac{0 \equiv 1 : \mathbb{I}}{\mathcal{J}}$$

   Idea: A term $a : \prod_{i:\mathbb{I}} A(i)$ is an *identification* between $a(0)$ and $a(1)$.

2. Universe $\mathbb{F}$ of propositions closed under at least:
   - extensional equality $(r =_{\mathbb{I}} s)$ of dimensions $r, s : \mathbb{I}$
   - conjunction $\phi \wedge \psi$ and (extensional) disjunction $\phi \vee \psi$
   - universal quantification over the interval $\forall i : \mathbb{I}.\phi(i)$

   Idea: A term $a : \phi \to A$ is a *partial element* of A, defined only when ϕ is true.

New computations! :-)

Cubical type theory extends MLTT with new generic functions

$$\frac{A : \mathbb{I} \to \mathcal{U} \qquad \phi : \mathbb{F} \qquad r, s : \mathbb{I} \qquad f : \prod_{i:\mathbb{I}} \prod_{p:(i =_{\mathbb{I}} r) \vee \phi} A(i)}{\mathbf{com}_A^{r \rightsquigarrow s}(f) : \{\tilde{f}_s : A(s) \mid \forall p. \tilde{f}_s = f(s, p)\}}$$

Coercion/transport, symmetry, and transitivity are all special cases of com_A.

Resulting theory of equality much easier to use than ITT+J!

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Canonicity at the limits of operational tractability: normalization will require new techniques (this thesis!).

New computations! (-:

Simpler alternative to operational semantics + PERs with coherent expansion: **equational theory + Artin gluing**, as proposed by Awodey in 2015.

> S., Angiuli, Gratzer (2019). **"Cubical Syntax for Reflection-Free Extensional Equality."** FSCD 2019.

S., Angiuli, and Gratzer [SAG19] present an **easy and complete** proof of canonicity for a version of cubical type theory in less than 30 pages. *Trial run of the "objective metatheory."*

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3. **proof relevance:**
   - “property of raw syntax” $\Longrightarrow$ “structure over objective syntax”
   - generalization proof-relevant logical relations is forced!

Local invariance of objective syntax

Syntax of type theory with function types expressed as an equational LF signature:[1]

$$\mathbf{Tp} : \mathbf{Kind} \qquad \mathbf{Tm} : \mathbf{Tp} \to \mathbf{Type} \qquad \mathbf{Fn} : \mathbf{Tp} \times \mathbf{Tp} \to \mathbf{Tp}$$

$$\alpha_{\mathbf{Fn}} : \textstyle\prod_{A,B:\mathbf{Tp}} \mathbf{Tm}(\mathbf{Fn}(A,B)) \cong (\mathbf{Tm}(A) \to \mathbf{Tm}(B))$$

Above: introduction, elimination, computation, and uniqueness rules bundled in $\alpha_{\mathbf{Fn}}$.

Local invariance: impossible to utter a distinction between judgmentally equal terms. (Anti-bureaucratic power move!)

[1] Equational LF due to Uemura [Uem19] with universes $\mathbf{Type} \subseteq \mathbf{Kind}$; kinds closed under dependent products along types as in Harper, Honsell, and Plotkin [HHP93].

Global invariance of objective syntax

$$\mathbf{Tp} : \mathbf{Kind} \qquad \mathbf{Tm} : \mathbf{Tp} \to \mathbf{Type} \qquad \mathbf{Fn} : \mathbf{Tp} \times \mathbf{Tp} \to \mathbf{Tp}$$
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The signature Σ above involves a *choice* of LF encoding, but metatheorems don't depend on how we set up the function type (which is uniquely determined up to iso).

Global invariance: Σ presents a *classifying category* $\mathcal{C}_\Sigma$ of judgments and deductions, which we work with *up to weak equivalence of categories*.

Proof relevance in the objective metatheory

Let’s remember how logical relations work…

(Unary) logical relations on closed terms is: for each sort A *sort*$_\Sigma$, a subset $\tilde{A} \subseteq \{a \mid \cdot \vdash_\Sigma a : A\}/\equiv_\Sigma$ respecting all the operations of Σ.

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Thinking of subsets as injective functions:

$$\begin{tikzcd}
\tilde{B} \arrow[r] \arrow[d, hook] & \tilde{A} \arrow[d, hook] \\
\rho(B) \arrow[r, "\rho(a)"'] & \rho(A)
\end{tikzcd}$$

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Fundamental Theorem of Logical Relations: prove that $\mathcal{G}_\Sigma$ is a model of the theory Σ!

FTLR Idea: interpret each type into $\mathcal{G}_\Sigma$ in such a way that an element carries the proof of the desired metatheorem, e.g. canonicity at base type:

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\mathcal{C}_\Sigma \arrow[r, "\rho"'] & \mathbf{Set}
\end{tikzcd}
\qquad
\begin{tikzcd}
\{\mathbf{yes}, \mathbf{no}\} \arrow[d, hook, "\mathcal{G}_\Sigma[\![\mathbf{ans}]\!]"] \\
\rho(\mathbf{ans})
\end{tikzcd}$$

Connectives that have β/η laws are uniquely determined! No need (or ability) to be clever.

Proof-relevant logical relations!

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\sum_{A:\rho(\mathbf{Tp})} (\rho(\mathbf{Tm})(A) \to \mathbf{Set}_{small}) \arrow[d, "\mathcal{G}_\Sigma[\![\mathbf{Tp}]\!]"] \\
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Abstract Artin gluing: logical relations as types!

Constructing the logical relation in $\mathcal{G}_\Sigma$ over $\mathcal{C}_\Sigma$ still very technical! **Trivialized by “synthetic Tait computability” (STC).**[3]

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Facts/Axioms:

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3. A logical relation can be realigned to have different (but isomorphic) syntactic part [OP16].

Idea: axiomatize a Σ-algebra $\mathcal{A}^{\circ}_{\Sigma}$ in the $\bigcirc$-modal fragment of **STC**, then construct a Σ-algebra $\mathcal{A}^{*}_{\Sigma}$ in **STC** such that $\bigcirc(\mathcal{A}^{*} = \mathcal{A}^{\circ})$ holds.

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Payoff: painful construction of (e.g.) dependent product in logical relations made trivial, because open modalities commute with dependent products. *No more technical lemmas!*

Show $\bigcirc\mathbf{Tp}^* \cong \mathbf{Tp}^\circ$.

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$$\mathbf{Tp}^\circ$$

$\square$

Closure under dependent products.

Construct $\Pi^* : (A : \mathbf{Tp}^*, B : A \to \mathbf{Tp}^*) \to \mathbf{Tp}^*$ with $\bigcirc \Pi^*(A, B) = \Pi^\circ(A^\circ, \lambda x. B^\circ(x))$.

Closure under dependent products.

Fix $A : \mathbf{Tp}^*, B : A \to \mathbf{Tp}^*$ to construct $\Pi^*(A, B)$ with syntactic part $\Pi^\circ(A^\circ, \lambda x.B^\circ(x))$.

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$\square$

S. & Harper introduced **STC** to give a trivial proof of a non-trivial parametricity result for effectful ML modules:

> S., Harper (2020). **“Logical Relations as Types: Proof-Relevant Parametricity for Program Modules”**. Under review.

This thesis: use **STC** to prove normalization of Cartesian cubical type theory / $\mathbf{TT}_{\square}$.

Proposed work

Prove normalization for Cartesian cubical type theory ($\mathrm{TT}_{\square}$).

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3. That does it! **Next: actually construct the model.**

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- $\widehat{\mathcal{R}}$ has monoidal closed structure $(\otimes, \multimap)$ to implement *bunches* of non-equal dimensions and fresh binders:

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Foreseen difficulties: What to do with the inconsistent context $(\Gamma, p : 0 =_{\mathbb{I}} 1)$?

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$\mathbf{Sh}(\mathcal{K})$ a suitable model of **STC**.

Modeling neutral/normal forms

From now on, we work in the language of **STC**. Let A be a type of $\textbf{TT}_{\square}$.

1. We have an object $A^{\mathbf{Var}} : \mathcal{U}$ of variables with $\bigcirc(A^{\mathbf{Var}} = A^{\circ})$
2. Extend it to a definition of *neutral forms* $A^{\mathbf{Ne}}$ of type A
3. Extend it to a definition of *normal forms* $A^{\mathbf{Nf}}$ of type A.

Subtleties: representation of interval dimension binders, tensors of dimensions.

The computability algebra

From now on, we work abstractly in the (extended) **STC** substantiated above. Inspired by and improving on Coquand [Coq19], we define the computability algebra of types:

$$\mathbf{Tp}^* \cong \sum \left\{ \begin{array}{l} A^\circ : \mathbf{Tp}^\circ \\ \mathfrak{A} : \left\{\mathfrak{A} : \mathbf{Tp}^{\mathbf{Nf}} \;\middle|\; \bigcirc(\mathfrak{A} = A^\circ)\right\} \\ A^* : \{A^* : \mathcal{U} \mid \bigcirc(A^* = \mathbf{Tm}^\circ(A^\circ))\} \\ \mathbf{reflect}_A : \left\{\mathsf{f} : A^{\mathbf{Ne}} \to A^* \;\middle|\; \bigcirc(\mathsf{f} = \mathbf{id})\right\} \\ \mathbf{reify}_A : \left\{\mathsf{f} : A^* \to A^{\mathbf{Nf}} \;\middle|\; \bigcirc(\mathsf{f} = \mathbf{id})\right\} \end{array} \right\}$$

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Foreseen difficulties: We must define a version of $\mathbf{Tp}^*$ that equips each *family* $\mathbb{I} \to \mathbf{Tp}^*$ with a composition operation; possible if the interval is atomic [Lic+18]. This is the main thing that could go wrong.

Timeline and fallback positions

The goal is a proof of normalization for cubical type theory with a univalent universe. Granular prediction impossible, but I estimate the following milestones to serve as fallback positions if part of this turns out to be intractable.

- **Now + 6 months:** a proof of β/η normalization for MLTT + $\mathbb{I}$ + $\mathbb{F}$ + extension types, no universes or HITs.
- **Now + 8 months:** a mathematical specification of the elaboration of a `redtt`/`cooltt`-style external language to the core type theory specified above.
- **Now + 12 months:** extension of proof to include a univalent universe.

Stretch goals

The following are things that I *do not* promise to do, but which might happen along the way if my progress is better than expected. If I don't do it, one of your students should!

- **Extensions of the `cooltt` implementation:** modules, higher inductive types, more sophisticated universe hierarchies, or support for **STC** modalities.
- **Extensions of synthetic Tait computability** to account for modal and substructural type theory.
- **Objective metatheory of effectful PLs:** Sterling and Harper [SH20] just a first step, more development needed!

References I

[AGV72] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Théorie des topos et cohomologie étale des schémas*. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J.-L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269, 270, 305. Berlin: Springer-Verlag, 1972.

[AHH17] Carlo Angiuli, Kuen-Bang Hou (Favonia), and Robert Harper. *Computational Higher Type Theory III: Univalent Universes and Exact Equality*. 2017. arXiv: 1712.01800.

[AHS95] Thorsten Altenkirch, Martin Hofmann, and Thomas Streicher. "Categorical reconstruction of a reduction free normalization proof". In: *Category Theory and Computer Science*. Ed. by David Pitt, David E. Rydeheard, and Peter Johnstone. Berlin, Heidelberg: Springer Berlin Heidelberg, 1995.

[AK16a] Thorsten Altenkirch and Ambrus Kaposi. "Normalisation by Evaluation for Dependent Types". In: *1st International Conference on Formal Structures for Computation and Deduction (FSCD 2016)*. Ed. by Delia Kesner and Brigitte Pientka. Vol. 52. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016. DOI: 10.4230/LIPIcs.FSCD.2016.6.

References II

[AK16b] Thorsten Altenkirch and Ambrus Kaposi. “Type Theory in Type Theory Using Quotient Inductive Types”. In: *Proceedings of the 43rd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*. POPL ’16. St. Petersburg, FL, USA: ACM, 2016. DOI: 10.1145/2837614.2837638.

[All87] Stuart Frazier Allen. “A non-type-theoretic semantics for type-theoretic language”. PhD thesis. Ithaca, NY, USA: Cornell University, 1987.

[Ang+17] Carlo Angiuli, Guillaume Brunerie, Thierry Coquand, Kuen-Bang Hou (Favonia), Robert Harper, and Daniel R. Licata. *Cartesian Cubical Type Theory*. Preprint. Dec. 2017.

[Ang+18a] Carlo Angiuli, Evan Cavallo, Kuen-Bang Hou (Favonia), Robert Harper, Anders Mörtberg, and Jonathan Sterling. redtt*: implementing Cartesian cubical type theory*. Dagstuhl Seminar 18341: Formalization of Mathematics in Type Theory. 2018.

[Ang+18b] Carlo Angiuli, Evan Cavallo, Kuen-Bang Hou (Favonia), Robert Harper, and Jonathan Sterling. “The RedPRL Proof Assistant (Invited Paper)”. In: *Proceedings of the 13th International Workshop on Logical Frameworks and Meta-Languages: Theory and Practice, LFMTP@FSCD 2018, Oxford, UK, 7th July 2018*. 2018. DOI: 10.4204/EPTCS.274.1.

References III

[Ang+19] Carlo Angiuli, Guillaume Brunerie, Thierry Coquand, Kuen-Bang Hou (Favonia), Robert Harper, and Daniel R. Licata. *Syntax and Models of Cartesian Cubical Type Theory*. Preprint. Feb. 2019.

[Ang19] Carlo Angiuli. "Computational Semantics of Cartesian Cubical Type Theory". PhD thesis. Carnegie Mellon University, 2019.

[AS19] Carlo Angiuli and Jonathan Sterling. "Lectures on type theory". Based on lectures given by Angiuli and Sterling in autumn of 2019. 2019.

[AW09] Steve Awodey and Michael A. Warren. "Homotopy theoretic models of identity types". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 146.1 (Jan. 2009). ISSN: 0305-0041. DOI: 10.1017/S0305004108001783.

[Awo18a] Steve Awodey. "A cubical model of homotopy type theory". In: *Annals of Pure and Applied Logic* 169.12 (2018). Logic Colloquium 2015. ISSN: 0168-0072. DOI: 10.1016/j.apal.2018.08.002.

[Awo18b] Steve Awodey. "Natural models of homotopy type theory". In: *Mathematical Structures in Computer Science* 28.2 (2018). DOI: 10.1017/S0960129516000268.

[Car78] John Cartmell. "Generalised Algebraic Theories and Contextual Categories". PhD thesis. Oxford University, Jan. 1978.

References IV

[CCD17] Simon Castellan, Pierre Clairambault, and Peter Dybjer. "Undecidability of Equality in the Free Locally Cartesian Closed Category (Extended version)". In: *Logical Methods in Computer Science* 13.4 (2017).

[CH19] Evan Cavallo and Robert Harper. "Higher Inductive Types in Cubical Computational Type Theory". In: *Proc. ACM Program. Lang.* 3.POPL (Jan. 2019). ISSN: 2475-1421. DOI: 10.1145/3290314.

[CHM18] Thierry Coquand, Simon Huber, and Anders Mörtberg. "On Higher Inductive Types in Cubical Type Theory". In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. Oxford, United Kingdom: ACM, 2018. DOI: 10.1145/3209108.3209197.

[Coh+15] Cyril Cohen, Thierry Coquand, Simon Huber, and Simon Mörtberg. `cubical`*: Implementation of Univalence in Cubical Sets*. 2015.

[Coh+17] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. "Cubical Type Theory: a constructive interpretation of the univalence axiom". In: *IfCoLog Journal of Logics and their Applications* 4.10 (Nov. 2017).

[Coh+18] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. `cubicaltt`*: Experimental implementation of Cubical Type Theory*. 2018.

References V

[Coq19] Thierry Coquand. "Canonicity and normalization for dependent type theory". In: *Theoretical Computer Science* 777 (2019). In memory of Maurice Nivat, a founding father of Theoretical Computer Science - Part I. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2019.01.015. arXiv: 1810.09367.

[Dyb96] Peter Dybjer. "Internal type theory". In: *Types for Proofs and Programs: International Workshop, TYPES '95 Torino, Italy, June 5–8, 1995 Selected Papers*. Ed. by Stefano Berardi and Mario Coppo. Berlin, Heidelberg: Springer Berlin Heidelberg, 1996.

[Fio02] Marcelo Fiore. "Semantic Analysis of Normalisation by Evaluation for Typed Lambda Calculus". In: *Proceedings of the 4th ACM SIGPLAN International Conference on Principles and Practice of Declarative Programming*. PPDP '02. Pittsburgh, PA, USA: ACM, 2002. DOI: 10.1145/571157.571161.

[Fio12] Marcelo Fiore. *Discrete generalised polynomial functors.* Slides from talk given at ICALP 2012. 2012.

[Fre78] Peter Freyd. "On proving that 1 is an indecomposable projective in various free categories". Unpublished manuscript. 1978.

[GSB19] Daniel Gratzer, Jonathan Sterling, and Lars Birkedal. "Implementing a Modal Dependent Type Theory". In: *Proceedings of the ACM on Programming Languages* 3.ICFP (July 2019). ISSN: 2475-1421. DOI: 10.1145/3341711.

References VI

[HHP93] Robert Harper, Furio Honsell, and Gordon Plotkin. “A Framework for Defining Logics”. In: *J. ACM* 40.1 (Jan. 1993). ISSN: 0004-5411. DOI: 10.1145/138027.138060.

[HS98] Martin Hofmann and Thomas Streicher. “The groupoid interpretation of type theory”. In: *Twenty-five years of constructive type theory (Venice, 1995)*. Vol. 36. Oxford Logic Guides. New York: Oxford Univ. Press, 1998.

[Hub18] Simon Huber. “Canonicity for Cubical Type Theory”. In: *Journal of Automated Reasoning* (June 13, 2018). ISSN: 1573-0670. DOI: 10.1007/s10817-018-9469-1.

[Joh02] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium: Volumes 1 and 2*. Oxford Logical Guides 43. Oxford Science Publications, 2002.

[Law63] F. William Lawvere. “Functorial Semantics of Algebraic Theories”. PhD thesis. Columbia University, 1963.

[Lic+18] Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. “Internal Universes in Models of Homotopy Type Theory”. In: *3rd International Conference on Formal Structures for Computation and Deduction, FSCD 2018, July 9-12, 2018, Oxford, UK*. 2018. DOI: 10.4230/LIPIcs.FSCD.2018.22.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

References VII

[MA18] Anders Mörtberg and Carlo Angiuli. yacctt*: Yet Another Cartesian Cubical Type Theory*. 2018.

[Mar79] Per Martin-Löf. "Constructive Mathematics and Computer Programming". In: *6th International Congress for Logic, Methodology and Philosophy of Science*. Published by North Holland, Amsterdam. 1982. Hanover, Aug. 1979.

[Mar84] Per Martin-Löf. *Intuitionistic type theory. Notes by Giovanni Sambin.* Vol. 1. Studies in Proof Theory. Bibliopolis, 1984.

[New18] Clive Newstead. "Algebraic Models of Dependent Type Theory". PhD thesis. Carnegie Mellon University, 2018.

[NPS90] Bengt Nordström, Kent Peterson, and Jan M. Smith. *Programming in Martin-Löf's Type Theory*. Vol. 7. International Series of Monographs on Computer Science. NY: Oxford University Press, 1990.

[OP16] Ian Orton and Andrew M. Pitts. "Axioms for Modelling Cubical Type Theory in a Topos". In: *25th EACSL Annual Conference on Computer Science Logic, CSL 2016, August 29 - September 1, 2016, Marseille, France*. 2016.

[Red18] The RedPRL Development Team. redtt. 2018.

[Red20] The RedPRL Development Team. cooltt. 2020.

References VIII

[RSS17] Egbert Rijke, Michael Shulman, and Bas Spitters. *Modalities in homotopy type theory*. 2017. arXiv: 1706.07526.

[SA20] Jonathan Sterling and Carlo Angiuli. “Gluing models of type theory along flat functors”. Unpublished draft. 2020.

[SAG19] Jonathan Sterling, Carlo Angiuli, and Daniel Gratzer. “Cubical Syntax for Reflection-Free Extensional Equality”. In: *4th International Conference on Formal Structures for Computation and Deduction (FSCD 2019)*. Ed. by Herman Geuvers. Vol. 131. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2019. DOI: 10.4230/LIPIcs.FSCD.2019.31. arXiv: 1904.08562.

[SAG20] Jonathan Sterling, Carlo Angiuli, and Daniel Gratzer. *A cubical language for Bishop sets*. 2020. arXiv: 2003.01491.

[SG20] Jonathan Sterling and Daniel Gratzer. “Lectures on Synthetic Tait Computability”. Based on lectures given by Sterling in the summer of 2020. 2020.

[SH18] Jonathan Sterling and Robert Harper. “Guarded Computational Type Theory”. In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. Oxford, United Kingdom: ACM, 2018. arXiv: 1804.09098.

References IX

[SH20] Jonathan Sterling and Robert Harper. “Logical Relations As Types: Proof-Relevant Parametricity for Program Modules”. Unpublished draft. 2020.

[Shu11] Michael Shulman. *Internalizing the External, or The Joys of Codiscreteness*. blog. 2011.

[Shu13] Michael Shulman. *Scones, Logical Relations, and Parametricity*. Blog. 2013.

[Shu15] Michael Shulman. “Univalence for inverse diagrams and homotopy canonicity”. In: *Mathematical Structures in Computer Science* 25.5 (2015). DOI: 10.1017/S0960129514000565.

[SS18] Jonathan Sterling and Bas Spitters. *Normalization by gluing for free λ-theories*. Sept. 2018. arXiv: 1809.08646 [cs.LO].

[Ste20] Jonathan Sterling. *Objective Metatheory of Cubical Type Theories*. Thesis Proposal. 2020.

[Str91] Thomas Streicher. *Semantics of Type Theory: Correctness, Completeness, and Independence Results*. Cambridge, MA, USA: Birkhauser Boston Inc., 1991.

References X

[Str98] Thomas Streicher. "Categorical intuitions underlying semantic normalisation proofs". In: *Preliminary Proceedings of the APPSEM Workshop on Normalisation by Evaluation*. Ed. by O. Danvy and P. Dybjer. Department of Computer Science, Aarhus University, 1998.

[Tai67] W. W. Tait. "Intensional Interpretations of Functionals of Finite Type I". In: *The Journal of Symbolic Logic* 32.2 (1967). ISSN: 00224812.

[Uem19] Taichi Uemura. *A General Framework for the Semantics of Type Theory*. 2019. arXiv: 1904.04097.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[VMA19] Andrea Vezzosi, Anders Mörtberg, and Andreas Abel. "Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types". In: *Proceedings of the 24th ACM SIGPLAN International Conference on Functional Programming*. ICFP '19. Boston, Massachusetts, USA: ACM, 2019. DOI: 10.1145/3341691.