First Steps in Synthetic Tait Computability

The Objective Metatheory of Cubical Type Theory

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To my mother, LeeAnn.

Heartfelt thanks to Bob for making this whole experience possible!

Dependent type theory is...

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a language for math

Dependent type theory is...

a language for homotopical math

Dependent type theory is...

a programming language

Dependent type theory is...

a programming language + program logic

Dependent type theory is...

a metalanguage for PL syntax

Dependent type theory is...

a metalanguage for PL semantics

Requirements of type theoretic tools

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Semantic properties

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(all models)

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Swedish philosophy

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general mathematics

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- **Agda, Coq, Lean:** Martin-Löf identity type does not derive funext
- **Nuprl, Agda, Coq, Lean:** 1-dimensional equality incompatible with univalence

What we've been working on

Our aim has been to achieve all goals at once; HoTT achieves the semantic goals, but it is not a PL. Cubical type theory[1] designed to reconcile all these constraints.

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This dissertation proves that the type theories underlying both `redtt` and Cubical Agda have **decidable type checking**. The main ingredient is a new technique called ***synthetic Tait computability*** (STC) abstracting Artin gluing and logical relations.

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1. Cubical type theory

What is cubical type theory / □TT?

□TT is an extension of Martin-Löf's Type Theory by an interval:

- a new sort $\Gamma \vdash \mathbb{I}$ and context extension $\Gamma, i : \mathbb{I}$
- with endpoints $\Gamma \vdash 0, 1 : \mathbb{I}$

Why? A new way to think about equality (paths) as *figures* of shape $\mathbb{I}$.

$$(a_0 =_A a_1) :\equiv \{p : \mathbb{I} \to A \mid p(0) \equiv a_0 \wedge p(1) \equiv a_1\}$$

Supports function extensionality, type extensionality (univalence), and effective quotients like Homotopy Type Theory/HoTT,[2] but has stronger syntactic/computational properties.

[2] Univalent Foundations Program (2013)

Computation in □TT: prior art

The state of the art (Huber, 2018; Angiuli, Hou (Favonia), and Harper, 2018):

Theorem (Cubical canonicity)

If $\vec{\imath} : \mathbb{I}^n \vdash M(\vec{\imath}) : \mathsf{bool}$ *is a closed n-cube of booleans, then either* $\vec{\imath} : \mathbb{I}^n \vdash M(\vec{\imath}) \equiv \mathsf{tt} : \mathsf{bool}$ *or* $\vec{\imath} : \mathbb{I}^n \vdash M(\vec{\imath}) \equiv \mathsf{ff} : \mathsf{bool}$.

Hence **□TT** is programming language.

Cubical canonicity is only about computation of closed *n*-cubes.
But **implementation** (type checking, elaboration) requires computation in *arbitrary* contexts Γ, *i.e.* normalization.

Results of this dissertation

I have proved the following suite of results for **$\square$TT** with a countable cumulative hierarchy of universes:[3]

Theorem (Normalization)

*There is a computable function assigning to every type $\Gamma \vdash A$ and every term $\Gamma \vdash a : A$ of **$\square$TT** a unique normal form.*

Corollary (Decidability of equality)

*Judgmental equality $\Gamma \vdash A \equiv B$ and $\Gamma \vdash a \equiv b : A$ in **$\square$TT** is decidable.*

Corollary (Injectivity of type constructors)

If $\Gamma \vdash \Pi(A, B) \equiv \Pi(A', B')$ then $\Gamma \vdash A \equiv A'$ and $\Gamma, x : A \vdash B(x) \equiv B'(x)$.

[3] The preliminary result for **$\square$TT** without universes is j.w.w. Angiuli published in LICS'21 (Sterling and Angiuli, 2021).

2. Synthetic Tait computability

Proving metatheorems using Tait's method

In 1967, Tait introduced his *method of computability*[4]; Tait computability has remained our only scalable tool for proving metatheorems for logics and type theory (canonicity, normalization, parametricity, conservativity, *etc.*).[5]

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Idea: an "interpretation" that equips each type A with an predicate $[\![A]\!]$ on elements of A; then show that all *terms* preserve the predicates.

1. First choose the predicate at base type to make soundness of the interpretation imply the desired metatheorem.
2. Then "draw the rest of the owl".

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Operational Tait computability

First define operational semantics $\mapsto^*$ on raw closed terms.

Example (Canonicity)

To prove canonicity, we choose the following predicates:

$$\begin{aligned} [\![\mathsf{bool}]\!](b) &:= (b \mapsto^* \mathsf{tt} \vee b \mapsto^* \mathsf{ff}) \\ [\![A \to B]\!](f) &:= (\forall x : A. [\![A]\!](x) \to [\![B]\!](f(x))) \end{aligned}$$

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(None of the above have satisfactory answers in operational Tait computability.)

The outer limits of operational Tait computability

Specifying and verifying the domain and closure conditions of computability predicates for *cubical* ***canonicity*** proved nearly intractable, *pace* Huber (2018) and Angiuli, Hou (Favonia), and Harper (2018).

Motivated S., Angiuli, and Gratzer to pursue an *algebraic*/gluing-based version of Tait computability for **□TT**[6] à la Coquand (2018), as suggested by Awodey.

Idea: work only with *quotiented* typed terms, make computability predicates proof-relevant. **Outcome:** all difficulties disappeared for cubical canonicity, normalization still required fundamentally new ideas (this dissertation).

Synthetic Tait computability = type theoretic abstraction of the algebraic gluing argument à la Orton and Pitts (2016).

[6]Sterling, Angiuli, and Gratzer (2019)

Introducing synthetic Tait computability

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STC abstracts logical relations by isolating the relationship between syntax and semantics as a pair of modalities.[7]

Expressive enough to recover and simplify existing LR arguments. **More importantly**, STC gave me new geometrical intuitions that I used to solve cubical normalization.

[7](For experts: STC is the internal language of topoi equipped with open/closed partitions.)

Mixing syntax and semantics

What is really going on in Tait computability? We are *immersing* syntax in a more powerful language (the language of computability predicates) that can express the semantic invariants we want.

(Smoother to develop and use if we generalize to **computability *structures***, *i.e.* **proof-relevant** computability predicates.[8])

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e.g. the computability structure of the booleans:

$$[\![\mathsf{bool}]\!] := \left(x : \boxed{\mathsf{bool}}\right) \times \boxed{\boxed{x = \mathsf{tt}} + \boxed{x = \mathsf{ff}}}$$

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Piecing together syntax and semantics

Computability structures built from syntax and semantics.

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Computability structures built from $\boxed{\text{syntax}}$ and $\boxed{\text{semantics}}$. These can be mixed and matched, but the satisfy some laws:

- Both $\boxed{-}$ and $\boxed{-}$ are *lex idempotent monads.*[9]

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- **Fracture:** any computability structure A can be reconstructed from $\colorbox{blue}{$A$}$, $\colorbox{magenta}{$A$}$, and $\colorbox{magenta}{$\colorbox{blue}{A}$}$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \colorbox{blue}{A} \arrow[d] \\
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The language of synthetic Tait computability

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Equivalently, extend type theory by a generic proposition $¶ : \mathbf{Prop}$ and define $\boxed{A} := A^{¶}$ and $\boxed{A} := A \cup_{A \times ¶} ¶$.

Internal language of topoi formed by *Artin gluing* (Artin, Grothendieck, and Verdier, 1972; Wraith, 1974; Rijke, Shulman, and Spitters, 2020).

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3. From the general to the particular. . .

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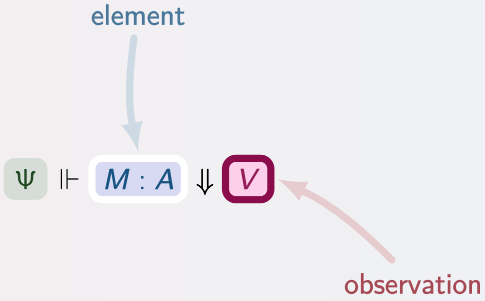

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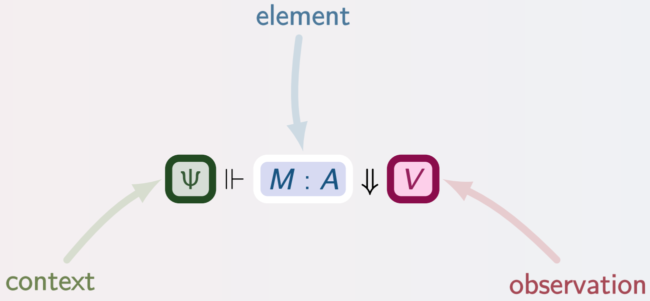

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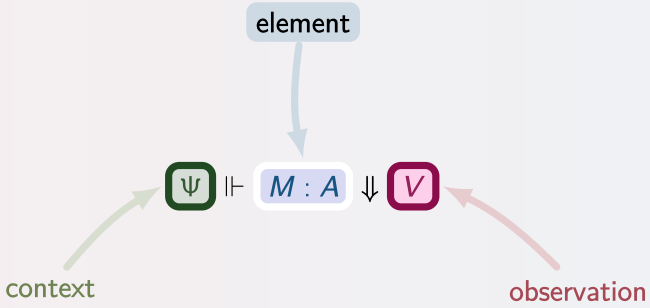

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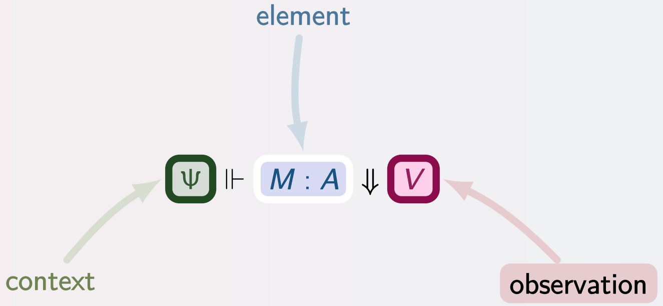

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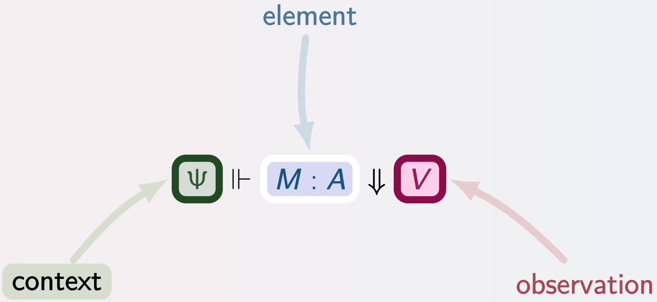

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canonicity: $\Gamma \in \{\cdot\}$; **cubical canonicity:** $\Gamma \in \{\mathbb{I}^n \mid n \in \mathbb{N}\}$; **normalization:** $\Gamma \in \{\vdash \mathit{ctx}\}$

Stability (or lack thereof) of observation

$$\boxed{x : \mathsf{nat}} \Vdash \boxed{x : \mathsf{nat}} \Downarrow \boxed{\mathbf{var}(x)}$$

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$$\boxed{\cdot} \xrightarrow{[\mathsf{fib}(9)/x]} \boxed{x : \mathsf{nat}} \Vdash \boxed{x : \mathsf{nat}} \Downarrow \boxed{\mathbf{var}(x)}$$

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$$\begin{tikzcd}
\boxed{\cdot} \arrow[r, "{[\mathsf{fib}(9)/x]}"] & \boxed{x : \mathsf{nat}} \arrow[r, phantom, "\Vdash"] & \boxed{x : \mathsf{nat}} \arrow[r, phantom, "\Downarrow"] \arrow[d, rightsquigarrow] & \boxed{\mathbf{var}(x)} \\
 & & \boxed{\mathsf{fib}(9) : \mathsf{nat}} &
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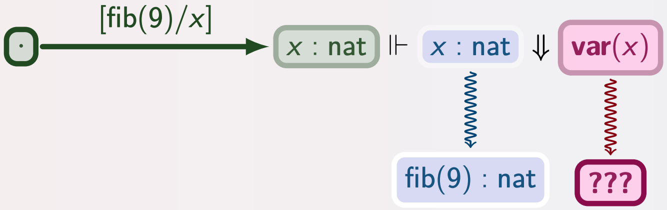
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Therefore normalization takes place over the category $\mathcal{R}$ of contexts and *structural renamings* (weakening, swapping, contraction).

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Unfortunately, just removing the substitutions for which neutral observations are unstable is not practicable for **□TT**. The problem lies with the interval:

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\end{tikzcd}$$

We shouldn't remove $[0/i], [1/i]$ from the category of contexts and renamings because we need $\mathbb{I}$ to restrict to something *representable* in $\mathrm{Pr}(\mathcal{R})$, *c.f.* **tininess** criterion (Licata, Orton, Pitts, and Spitters, 2018).

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Thesis: neutrals need to have a cubical substitution action (tininess of $\mathbb{I}$).

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Therefore we define an inductive family $\mathbf{Ne}_\phi(A)$ with $\boxed{\mathbf{Ne}_\phi(A)} \cong \boxed{A}$ comprised of neutrals e with $\partial e = \phi$. Traditional neutrals $\mathbf{Ne}_\bot(A)$; to model destabilization, $\mathbf{Ne}_\top(A) \cong \boxed{A}$.

Normalization via Tait's yoga

Tait (1967) introduced the famous *saturation yoga* for normalization:

$$\mathbf{Ne}(A) \subseteq [\![A]\!] \subseteq \mathbf{Nf}(A)$$

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\mathbf{Ne}(A) \arrow[r, "\uparrow_A"] \arrow[dr] & \llbracket A \rrbracket \arrow[r, "\downarrow_A"] \arrow[d] & \mathbf{Nf}(A) \arrow[dl] \\
& A &
\end{tikzcd}$$

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Yogic injury: unstable neutrals

$$\begin{tikzcd}
\mathbf{Ne}_{\phi}(A) \arrow[r, "\uparrow_A"] \arrow[dr] & \llbracket A \rrbracket \arrow[r, "\downarrow_A"] \arrow[d] & \mathbf{Nf}(A) \arrow[dl] \\
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 & A &
\end{tikzcd}$$

What if $\phi = \top$?

Yogic injury: unstable neutrals

$$\begin{tikzcd}
\mathbf{Ne}_{\top}(A) \arrow[r, "\uparrow_A"] \arrow[dr] & [\![A]\!] \arrow[r, "\downarrow_A"] \arrow[d] & \mathbf{Nf}(A) \arrow[dl] \\
& A &
\end{tikzcd}$$

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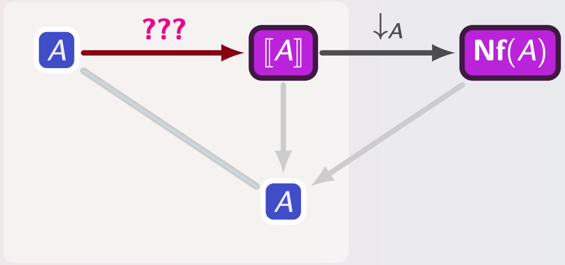

$$\begin{tikzcd}
A \arrow[r, "???"] \arrow[dr, no head] & \llbracket A \rrbracket \arrow[r, "\downarrow_A"] \arrow[d] & \mathbf{Nf}(A) \arrow[dl] \\
& A &
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What if $\phi = \top$? We must strengthen the "induction hypothesis".

Stabilization of neutrals

To strengthen the Tait reflection hypothesis, we **glue** unstable neutrals together with compatible computability data along their frontiers of instability.

$$\begin{tikzcd}
\mathbf{Ne}_\phi(A) \wr_\phi [\![A]\!] \arrow[r, dotted] \arrow[dd, dotted] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathbf{Ne}_\phi(A) \arrow[d] \\
 & A \arrow[d] \\
\phi \Rightarrow [\![A]\!] \arrow[r] & \phi \Rightarrow A
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“stabilized neutrals”

A spectrum of computability data

$$\begin{tikzcd}
\mathbf{Ne}_\perp(A) \arrow[r, rightsquigarrow, "\mathbf{Ne}_\phi \wr_\phi \llbracket A \rrbracket"] \arrow[d, mapsto] & \llbracket A \rrbracket \arrow[d, mapsto] \\
\perp \arrow[r, rightsquigarrow, "\phi"] & \top
\end{tikzcd}$$

Stabilization **interpolates** between neutrals and computability data.

A spectrum of computability data

“conventional neutrals”

$$\begin{tikzcd}
\mathbf{Ne}_\perp(A) \arrow[r, rightsquigarrow, "\mathbf{Ne}_\phi \wr_\phi [\![A]\!]"] \arrow[d, mapsto] & {[\![A]\!]} \arrow[d, mapsto] \\
\perp \arrow[r, rightsquigarrow, "\phi"] & \top
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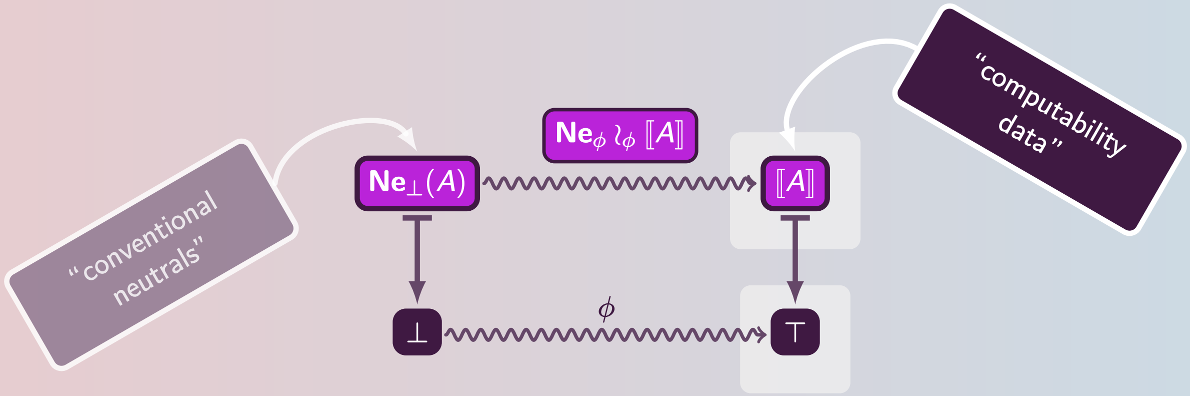

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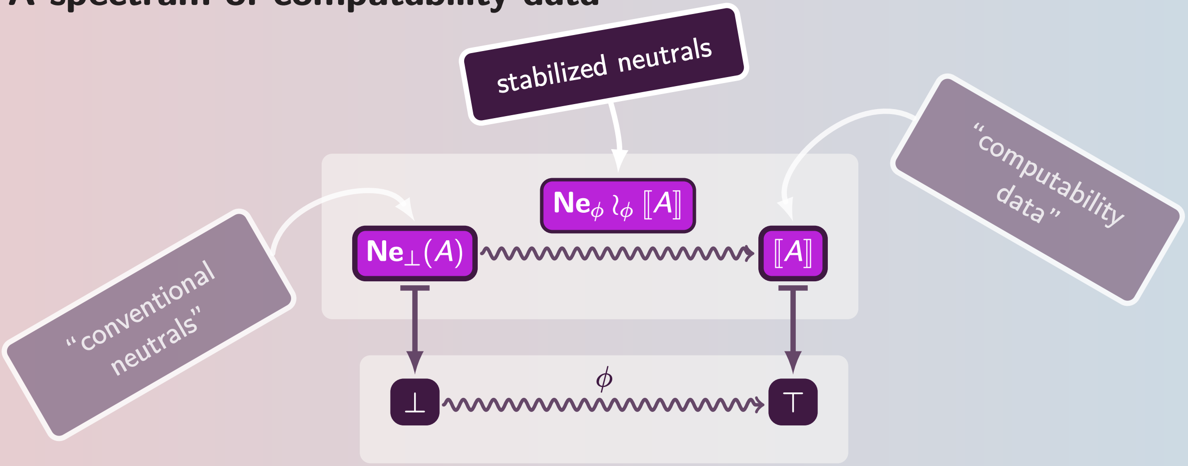

$$\begin{tikzcd}
& \mathbf{Ne}_{\phi} \wr_{\phi} [\![A]\!] & \\
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The stabilized Tait yoga

$$\begin{tikzcd}
 & & \phi \Rightarrow \llbracket A \rrbracket & & \\
\mathbf{Ne}_{\phi}(A) \wr_{\phi} \llbracket A \rrbracket \arrow[urr] \arrow[rr, "\uparrow_{A}^{\phi}"] \arrow[d] \arrow[drr] & & \llbracket A \rrbracket \arrow[u] \arrow[rr, "\downarrow_{A}"] \arrow[d] & & \mathbf{Nf}(A) \arrow[dll] \\
\mathbf{Ne}_{\phi}(A) \arrow[rr] & & A & &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & \phi \Rightarrow [\![A]\!] & \\
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Lemma (Saturation)

Every type of **□TT** *is closed under the* ***stabilized*** *Tait yoga.*

Summary of results

Lemma (Saturation)

Every type of **□TT** *is closed under the* ***stabilized*** *Tait yoga.*

The above is employed to obtain our main results:

Theorem (Normalization)

There is a computable function assigning to every type $\Gamma \vdash A$ *and every term* $\Gamma \vdash a : A$ *of* **□TT** *a unique normal form.*

Corollary (Decidability of equality)

Judgmental equality $\Gamma \vdash A \equiv B$ *and* $\Gamma \vdash a \equiv b : A$ *in* **□TT** *is decidable.*

Corollary (Injectivity of type constructors)

If $\Gamma \vdash \Pi(A, B) \equiv \Pi(A', B')$ *then* $\Gamma \vdash A \equiv A'$ *and* $\Gamma, x : A \vdash B(x) \equiv B'(x)$.

4. Taking stock

A computational conspectus on cubes. . .

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4. **computational interpretation of open terms**
   by Sterling and Angiuli (2021), this dissertation.

What's next for cubical type theory?

We have done more than enough cubical type *theory*. Time for applications!

- **applications to programming and verification**
  Cavallo and Harper (2020), Angiuli, Cavallo, Mörtberg, and Zeuner (2021), and Kidney and Wu (2021)
- **applications to denotational semantics**
  Møgelberg and Veltri (2019), Veltri and Vezzosi (2020), and Møgelberg and Vezzosi (2021)
- **applications to ordinary mathematics**
  Forsberg, Xu, and Ghani (2020)
- **applications to synthetic homotopy theory**
  Mörtberg and Pujet (2020), Cavallo (2021), and Brunerie, Ljungström, and Mörtberg (2021)

The era of synthetic Tait computability

- [LICS'21] **Normalization for cubical type theory** (Sterling and Angiuli, 2021)
- [JACM] **Logical Relations As Types: Proof-Relevant Parametricity for Program Modules** (Sterling and Harper, 2021)
- **Normalization for multi-modal type theory** (Gratzer, 2021).
- **A cost-aware logical framework** (Niu, Sterling, Grodin, and Harper, 2021)

Let a hundred phase distinctions bloom!

STC also leads to new perspectives on classic PL problems, *cf.* S. and Harper's analysis of the static/dynamic **phase distinction** and sealing in terms of STC.

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resource analysis	behavior	complexity
security / IFC	public	classified

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Thanks!

- Part I — syntax of dependent type theory *c.* 2021
- Part II — mathematical background (topos theory, universes)
- Part III — synthetic Tait computability, synthetic normalization for MLTT
- Part IV — cubical type theory, all main theorems

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