Normalization for Cubical Type Theory

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\(\text{\textcircled{TT}}\): cubical type theory

\(\text{\textcircled{TT}}\) is an extension of Martin-Löf’s Intensional Type Theory by an interval:
- a new sort \(\Gamma \vdash I\) and context extension \(\Gamma, i : \Pi \rightarrow \Gamma\)
- with endpoints \(\Gamma \vdash 0, 1 : \Pi\)
- and potentially further structure: \(r \sqcup s, r \sqcap s, \sim r\) [Coh+17]

**Why?** A new way to think about equality (paths) as figures of shape \(\Pi\).

\[
(a =_A b) := \{p : \Pi \rightarrow A \mid p(0) \equiv a \land p(1) \equiv b\}
\]

Supports function extensionality, type extensionality (univalence), and effective quotients.
Unlike HoTT, cubical type theory has good computational properties.

**Theorem (Cubical canonicity [AFH18; Hub18])**

If $\mathbb{I}^n \vdash M : \text{bool}$ is a closed $n$-cube of booleans, then either $\mathbb{I}^n \vdash M \equiv \text{tt} : \text{bool}$ or $\mathbb{I}^n \vdash M \equiv \text{ff} : \text{bool}$.

Therefore □TT can be used as a programming language [Ang+21], and we have multiple implementations, e.g. Cubical Agda, redtt, cooltt [Red18; Red20; VMA19].

**Canonicity** is only about computation in *purely cubical* contexts $i, j, k : \mathbb{I}$. **Implementation** requires computation in *arbitrary* contexts, *i.e.* normalization.
In what contexts do we compute?

\[ \Psi \vdash M : A \Downarrow \Gamma \]

**canonicity:**
- \( A \in \{ \text{nat} \} \)
- \( V \in \mathbb{N} \)

**normalization:**
- \( A \in \{ \Psi \vdash \text{type} \} \)
- \( V \in \{ \Psi \vdash \beta\eta \text{nf} A \} \)
- \( \Gamma \in \{ \vdash \text{ctx} \} \)

**cubical canonicity:**
- \( \Gamma \in \{ I_{\text{in}} \mid n \in \mathbb{N} \} \)
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- \( A \in \{ \text{nat} \} \)
- \( V \in N \)
- \( \Gamma \in \{ \cdot \} \)
- Cubical canonicity: \( \Gamma \in \{ \text{In} | n \in \mathbb{N} \} \)

**normalization:**
- \( A \in \{ \Psi \vdash \text{type} \} \)
- \( V \in \{ \Psi \vdash \text{\(\beta\eta\)}\text{nf} A \} \)
- \( \Gamma \in \{ \vdash \text{ctx} \} \)
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\[ \psi \vdash M : A \downarrow V \]

canonicity: \( A \in \{ \text{nat} \} \);

normalization: \( A \in \{ \psi \vdash \text{type} \} \);

\( \Gamma \in \{ \cdot \} \); cubical canonicity: \( \Gamma \in \{ I_n | n \in \mathbb{N} \} \);

\( \Gamma \in \{ \vdash \text{ctx} \} \);

element

context

observation
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**canonicity:** $\Gamma \in \{\cdot\}$; **cubical canonicity:** $\Gamma \in \{\Pi^n | n \in \mathbb{N}\}$; **normalization:** $\Gamma \in \{\vdash \text{ctx}\}$
Stability of observation *sans* $\parallel$

$$\frac{\textstyle x : \text{nat}}{\textstyle \downarrow \text{var}(x)}$$

In ITT, neutral observations (elimination forms blocked on variables) are closed under renaming, but not full substitution. Therefore, computation takes place in the Artin gluing of $\Pr(\text{C})$ and $\Pr(\text{R})$ where $\text{R} : \text{Cat}/\text{C}$ is the category of contexts and renamings (c.f. Kripke logical relations).
Stability of observation *sans* II

\[ \bullet \xrightarrow{\text{fib}(9)/x} x : \text{nat} \vdash x : \text{nat} \Downarrow \text{var}(x) \]
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\]

\[
\text{fib}(9) : \text{nat}
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\[
\begin{array}{c}
p : \text{fib} \Rightarrow_{\mathsf{nat}} \mathsf{nat} \quad i : \Box \\
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\]
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We cannot remove $[0/i]$ from the category of contexts and renamings because we need $I$ to restrict to something representable in $\text{Pr}(\text{R})$, c.f. tininess criterion [Lic+18].
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The power of dialectical thinking

**Thesis:** neutrals need to have a cubical substitution action.
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**Antithesis:** positive neutrality is not a cubical notion: under face maps \([\epsilon/i]\) a neutral observation can cease ‘being neutral’.
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**Synthesis:** the conditions under which a given neutral destabilizes are cubical. Given a neutral form $e : A$, write $\partial e$ for this *frontier of instability*. 
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Therefore, we define an inductive family \(\text{Ne}_\phi(A)\) of neutrals \(Tm(A)\) of neutrals with \(\partial e = \phi\). Traditional neutrals = \(\text{Ne}_\bot(A)\); to model destabilization, \(\text{Ne}_\top(A) \sim Tm(A)\).
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The standard **Tait yoga**

Tait [Tai67] introduced the famous *saturation yoga* for normalization.

![Diagram](attachment:diagram.png)
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![Diagram of Tait yoga](image)
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Stabilization of neutrals

Unstable neutrals are glued together with compatible computability data along their frontiers of instability.
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A spectrum of computability data

$\text{Ne}_{\phi \bot}([A])$

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$\bot$ $\phi$ $\top$

$\phi$ $\phi$ $[A]$

Stabilization interpolates between neutrals and computability data.
A spectrum of computability data stabilized neutrals

Ne_{\phi \wedge \phi} [A] 

\text{Ne}_{\perp} (A) \rightarrow [A] 

⊥ \rightarrow \phi \rightarrow T
A spectrum of computability data

$\perp$ (Ne)

$\perp \top \phi \lambda \phi [A]$

Stabilized neutrals

"conventional neutrals"

$\text{Ne}_\phi \lambda \phi [A]$

$\perp$

$\top$

$\phi$

$\perp$

$[A]$

$[A]$

$\top$

$\phi$

$\perp$
A spectrum of computability data

\[ \text{Ne}_{\varphi \land \varphi} [A] \]

stabilized neutrals

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The stabilized Tait yoga

\[ \phi \Rightarrow [A] \]

**Theorem.** Every type is closed under the stabilized Tait yoga.
The stabilized Tait yoga

\[ \phi \Rightarrow [A] \]

\[
\begin{align*}
\text{Ne}_\phi(A) \& \phi [A] & \mapsto [A] \\
\text{Ne}_\phi(A) & \mapsto Tm(A)
\end{align*}
\]

Theorem.
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The stabilized Tait yoga

\[ \text{Every type is closed under the stabilized Tait yoga.} \]
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\[ \text{Ne}_\phi(A) \sqsubseteq \phi \left[ A \right] \]

\[ \phi \Rightarrow \left[ A \right] \]

\[ \left[ A \right] \]

\[ \text{Tm}(A) \]

\[ \text{Nf}(A) \]

**Theorem.** Every type is closed under the **stabilized** Tait yoga.
Summary of results

For univalent \( \square \mathbb{TT} \) without universes, we have proved the following results:

1. Every type and every term has a *unique* normal form.
2. Judgmental equality of types and terms is decidable.
3. Type constructors (e.g. \( \Pi \)) are injective.
4. Type checking is decidable (corollary of 1–3).

**Forthcoming:** S. has extended this result to \( \square \mathbb{TT} \) with a countable hierarchy of univalent universes [Ste21].


References II


References III


