CONNECTIVES IN SEMANTICS OF TYPE THEORY

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Abstract. Some expository notes on the semantics of inductive types in Awodey's natural models [Awo18]. Many of the ideas explained are drawn from the work of Awodey [Awo18], Streicher [Str14], Gratzer, Kavvos, Nuyts, and Birkedal [Gra+20], and Sterling, Angiuli, and Gratzer [SAG20].

When is a model of type theory (i.e. a natural model) closed under a particular connective? And what is a general notion of connective? Letting $\mathcal{C}$ be a small category with a terminal object and $\mathcal{E} = \mathcal{P}\mathrm{r}(\mathcal{C})$ be its logos[1] of presheaves, a natural model is a representable natural transformation $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T} : \mathcal{E}$ [Awo18].

Connectives that commute with the Yoneda embedding $\mathcal{C} \overset{\mathrm{j}}{\hookrightarrow} \mathcal{E}$ (such as dependent product and sum) may be specified in a particularly simple way.

(1) First, one defines an endofunctor $\mathcal{E}^{\to}_{cart} \xrightarrow{\mathfrak{F}} \mathcal{E}^{\to}_{cart}$ taking a family (viewed as a universe) to the generic family for that connective. The intuition for this family is that the upstairs part carries the data of its introduction rule, and the downstairs part carries the data of its formation rule.

For instance, one may take $\mathfrak{F}$ to be the endofunctor that takes a map f to the functorial action of the polynomial endofunctor P_f on f itself. In this case, one has the notion of a dependent product type.

(2) Then, one requires a cartesian map $\mathfrak{F}(\boldsymbol{\tau}) \longrightarrow \boldsymbol{\tau}$, i.e. a pullback square of the following kind:

$$\begin{array}{ccc} \partial_0(\mathfrak{F}(\boldsymbol{\tau})) & \dashrightarrow & \dot{\mathbf{T}} \\ {\scriptstyle \mathfrak{F}(\boldsymbol{\tau})}\downarrow \lrcorner & & \downarrow{\scriptstyle \boldsymbol{\tau}} \\ \partial_1(\mathfrak{F}(\boldsymbol{\tau})) & \dashrightarrow & \mathbf{T} \end{array}$$

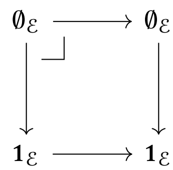
In the case of our example, the downstairs map becomes a *code* for the dependent product type, and the upstairs map becomes a constructor for λ-abstraction. The universal property of the pullback implements the elimination form (application) together with its computation and uniqueness rules.

1. Inductive types in the semantics of type theory

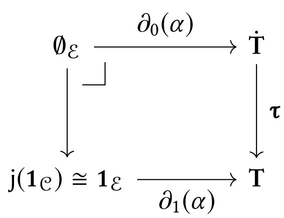
The account above, which works for essentially all connectives that can be understood by means of *mapping-in* universal properties, does not extend to either strict or weak inductive types; it is most common in the literature to account for these by means of internal orthogonality conditions and stable lifting structures respectively [Awo18; Gra+20; SAG20].

[1] I am following the convention of Anel and Joyal [AJ19] in referring to the formal algebraic dual of a topos as a *logos*, by analogy with frames and locales.

1.1. **Example: the empty coproduct.** Let us first gain some intuitions for why the naïve approach would fail to express the closure of $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$ under strict coproducts; it will suffice to consider the *empty* coproduct. We begin by defining an endofunctor $\mathfrak{F}_{\emptyset}$ on the cartesian arrow category $\mathcal{E}^{\rightarrow}_{cart}$. We will have $\mathfrak{F}_{\emptyset}$ take a map $A \xrightarrow{f} B$ to the universal map $\mathfrak{F}_{\emptyset}(f) = \emptyset_{\mathcal{E}} \longrightarrow \mathbf{1}_{\mathcal{E}}$; fixing $C \xrightarrow{g} D$ and a cartesian square $f \xrightarrow{\alpha} g$, we take $\mathfrak{F}_{\emptyset}(\alpha)$ to the following cartesian square:

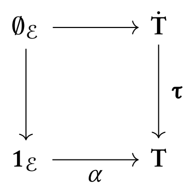


Naïvely, we might assume that the right thing to do next is ask for a cartesian map $\mathfrak{F}_{\emptyset}(\boldsymbol{\tau}) \xrightarrow{\alpha} \boldsymbol{\tau}$. But we can see that such a thing cannot exist; because $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$ is a representable map, the top-left corner of the following diagram must be representable:

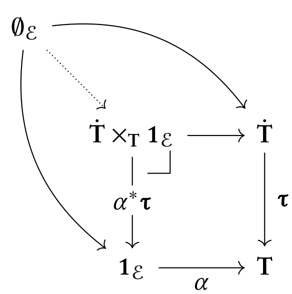


But the initial object of $\mathcal{E} = \mathcal{P}\mathrm{r}(\mathcal{C})$ cannot be representable: $\mathcal{E}$ is the *free* cocompletion. Generally speaking, for this approach to work, we would need the endofunctor $\mathfrak{F}$ to preserve representable maps, which $\mathfrak{F}_{\emptyset}$ clearly does not. We will show, however, an appropriate way to express the closure of $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$ under a strict empty type using orthogonality.

(1) First, we demand a code $\mathbf{1}_{\mathcal{E}} \xrightarrow{\alpha} \mathbf{T}$ (the formation rule).
(2) Then, we consider the following canonical (but not cartesian) square:



The *cartesian gap map* of this square is the homomorphism of spans induced by the actual pullback of $\boldsymbol{\tau}$ along α [Ane+17]:



(3) Then, we require that the cartesian gap map $\emptyset_{\mathcal{E}} \longrightarrow \boldsymbol{\tau}[\alpha]$ shall be *internally left orthogonal* to $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$ in $\mathcal{E}$. This means, for each $Z : \mathcal{E}$, a unique filler to any square of the following shape:

$$\begin{array}{ccc} Z \times \emptyset_{\mathcal{E}} & \longrightarrow & \dot{\mathbf{T}} \\ {\scriptstyle Z \times !}\downarrow & \overset{\exists!}{\nearrow} & \downarrow{\scriptstyle \boldsymbol{\tau}} \\ Z \times \dot{\mathbf{T}} \times_{\mathbf{T}} \mathbf{1}_{\mathcal{E}} & \longrightarrow & \mathbf{T} \end{array}$$

We have required a unique element of each type $Z \times \dot{\mathbf{T}} \times_{\mathbf{T}} \mathbf{1}_{\mathcal{E}} \xrightarrow{A} \mathbf{T}$; blowing up Z into a colimit of representables $\mathrm{j}(Z_i)$, we obtain exactly the intended notion: a unique element of any type in each inconsistent context.

1.2. **Strict connectives via orthogonality.** We fix an endofunctor $\mathcal{E}^{\to}_{cart} \xrightarrow{\mathfrak{F}} \mathcal{E}^{\to}_{cart}$; we will now describe what it takes for $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$ to be closed under the corresponding *strict* connective. For notational expediency, we will use the language of the codomain fibration $\mathcal{E}^{\to} \xrightarrow{\partial_1} \mathcal{E}$ rather than taking pullbacks in $\mathcal{E}$.

(1) First, we demand a commuting square $\mathfrak{F}(\boldsymbol{\tau}) \xrightarrow{\alpha} \boldsymbol{\tau} : \mathcal{E}^{\to}$, not necessarily cartesian. Here, $\partial_1(\alpha)$ is the "formation rule" and $\partial_0(\alpha)$ is the "introduction rule". What remains is to express the elimination rule (and its computation and uniqueness principles).
(2) We form the cartesian gap map for the diagram above:

$$\begin{array}{ccccc} \mathfrak{F}(\boldsymbol{\tau}) & \xrightarrow{\quad\alpha\quad} & & & \\ & \searrow{\scriptstyle \bar{\alpha}} & & & \\ & & \partial_1(\alpha)^*\boldsymbol{\tau} & \xrightarrow{\partial_1(\alpha)^\dagger\boldsymbol{\tau}} & \boldsymbol{\tau} \\ & & \downarrow & & \downarrow \\ & & \partial_1(\mathfrak{F}(\boldsymbol{\tau})) & \xrightarrow[\partial_1(\alpha)]{} & \mathbf{T} \end{array} \qquad \begin{array}{c} \mathcal{E}^{\to} \\ \downarrow \\ \mathcal{E} \end{array}$$

(3) For the elimination rule, we require that the gap map $\bar{\alpha}$ be internally left orthogonal $\partial_1(\mathfrak{F}(\boldsymbol{\tau}))^*\boldsymbol{\tau}$ in the slice category $\mathcal{C}_{/\partial_1(\mathfrak{F}(\boldsymbol{\tau}))}$.
(4) A large elimination may additionally be accomodated by requiring the gap map $\bar{\alpha}$ be internally left orthogonal to $\partial_1(\mathfrak{F}(\boldsymbol{\tau}))^*(\mathbf{T} \longrightarrow \mathbf{1}_{\mathcal{E}})$ in $\mathcal{C}_{/\partial_1(\mathfrak{F}(\boldsymbol{\tau}))}$.

Definition 1.1. Let $\mathcal{E}^{\to}_{cart} \xrightarrow{\mathfrak{F}} \mathcal{E}^{\to}_{cart}$ be a functor, and $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$ a natural model over $\mathcal{C}$, writing $\mathcal{E}$ for $\mathcal{P}\mathrm{r}(\mathcal{C})$. A *strict* $\mathfrak{F}$*-structure* for $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$ is a commuting square $\mathfrak{F}(\boldsymbol{\tau}) \xrightarrow{\alpha} \boldsymbol{\tau} : \mathcal{E}^{\to}$ such that the cartesian gap map $\mathfrak{F}(\boldsymbol{\tau}) \xrightarrow{\bar{\alpha}} \partial_1(\alpha)^*\boldsymbol{\tau}$ is is internally left orthogonal to $\partial_1(\mathfrak{F}(\boldsymbol{\tau}))^*\boldsymbol{\tau}$ in $\mathcal{C}_{/\partial_1(\mathfrak{F}(\boldsymbol{\tau}))}$. ❦

Example 1.2 (Strict coproducts)**.** We define an endofunctor $\mathcal{E}^{\to}_{cart} \xrightarrow{\mathfrak{F}_{\oplus}} \mathcal{E}^{\to}_{cart}$ capturing the generic binary coproduct situation for a given family. Given $A \xrightarrow{f} B$, we let $\partial_1(\mathfrak{F}_{\oplus}(f)) = B \times B$, to define the rest of the family $\mathfrak{F}_{\oplus}(f)$ to be the coproduct $\pi_2^* f + \pi_1^* f$ in $\mathcal{E}_{/B\times B}$.

Next, we fix $C \xrightarrow{g} D$ and a cartesian square $f \xrightarrow{\alpha} g$; we must exhibit a cartesian square $\mathfrak{F}_{\oplus}(f) \xrightarrow{\mathfrak{F}_{\oplus}(\alpha)} \mathfrak{F}_{\oplus}(g)$:

$$\begin{array}{ccc} \partial_0(\mathfrak{F}_{\oplus}(f)) & \dashrightarrow & \partial_0(\mathfrak{F}_{\oplus}(g)) \\ {\scriptstyle \pi_2^* f + \pi_1^* f}\downarrow & & \downarrow{\scriptstyle \pi_2^* g + \pi_1^* g} \\ B \times B & \dashrightarrow & C \times C \end{array}$$

Let $B \times B \xrightarrow{h} D \times D$ be the obvious map $\partial_1(\alpha) \times \partial_1(\alpha)$; rephrasing into the language of the codomain fibration, we may investigate the cartesian lift of this map:

$$\begin{array}{ccc} h^*(\pi_2^* g + \pi_1^* g) & \dashrightarrow & \pi_2^* g + \pi_1^* g \\ \downarrow & & \downarrow \\ B \times B & \xrightarrow[h]{} & D \times D \end{array}$$

By the universality of colimits, we may commute the pullback h^* into the coproduct:

$$\begin{array}{ccc} h^*\pi_2^* g + h^*\pi_1^* g & \longrightarrow & \pi_2^* g + \pi_1^* g \\ \downarrow & & \downarrow \\ B \times B & \xrightarrow[h]{} & D \times D \end{array}$$

It remains to show that $h^*\pi_i^* g \cong \pi_i^* f$. Because $\pi_i \circ h = \partial_1(\alpha) \circ \pi_i$, we may factor the cartesian map $h^*\pi_i^* g \longrightarrow g$ into the following composite of cartesian maps:

$$\begin{array}{ccccc} \pi_i^* f \cong h^*\pi_i^* g & \longrightarrow & f & \longrightarrow & g \\ \downarrow & & \downarrow & & \downarrow \\ B \times B & \longrightarrow & B & \longrightarrow & D \\ & \searrow & & \nearrow & \\ & & D \times D & & \end{array}$$

We have now defined the endofunctor $\mathcal{E}^{\rightarrow}_{cart} \xrightarrow{\mathfrak{F}_{\oplus}} \mathcal{E}^{\rightarrow}_{cart}$; given a natural model $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$, a strict $\mathfrak{F}_{\oplus}$-structure $\mathfrak{F}_{\oplus}(\boldsymbol{\tau}) \longrightarrow \boldsymbol{\tau}$ equips $\boldsymbol{\tau}$ with a coproduct connective, suitable introduction rules, and an elimination rule equipped with computation and uniqueness principles. ❦

1.3. **Comparing strict $\mathfrak{F}$-structures and cartesian squares.**

Lemma 1.3. *Let $\mathcal{E}^{\rightarrow}_{cart} \xrightarrow{\mathfrak{F}} \mathcal{E}^{\rightarrow}_{cart}$ be a functor, and let $\mathfrak{F}(\boldsymbol{\tau}) \xrightarrow{\alpha} \boldsymbol{\tau}$ be a cartesian map: then α is also a strict $\mathfrak{F}$-structure for $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$.*

Proof. Fixing $Z : \mathcal{E}_{/\partial_1(\mathfrak{F}(\boldsymbol{\tau}))}$, we must check that squares of the following kind have unique lifts:

$$\begin{array}{ccc} Z \times \mathfrak{F}(\boldsymbol{\tau}) & \xrightarrow{a} & \partial_1(\mathfrak{F}(\boldsymbol{\tau}))^* \dot{\mathbf{T}} \\ {\scriptstyle Z \times \bar{\alpha}}\downarrow & \overset{\exists!}{\nearrow} & \downarrow{\scriptstyle \partial_1(\mathfrak{F}(\boldsymbol{\tau}))^* \boldsymbol{\tau}} \\ Z \times \partial_1(\alpha)^* \boldsymbol{\tau} & \xrightarrow[A]{} & \partial_1(\mathfrak{F}(\boldsymbol{\tau}))^* \mathbf{T} \end{array}$$

Because α is cartesian, the vertical gap map $\bar{\alpha}$ is an isomorphism. Therefore, $Z \times \bar{\alpha}$ is also an isomorphism:

$$\begin{array}{ccc} Z \times \mathfrak{F}(\boldsymbol{\tau}) & \xrightarrow{\quad a \quad} & \partial_1(\mathfrak{F}(\boldsymbol{\tau}))^*\dot{\mathbf{T}} \\ {\scriptstyle \beta}\uparrow\; \Big\downarrow {\scriptstyle Z \times \bar{\alpha}} & \nearrow{\scriptstyle a \circ \beta} & \Big\downarrow {\scriptstyle \partial_1(\mathfrak{F}(\boldsymbol{\tau}))^*\boldsymbol{\tau}} \\ Z \times \partial_1(\alpha)^*\boldsymbol{\tau} & \xrightarrow[\quad A \quad]{} & \partial_1(\mathfrak{F}(\boldsymbol{\tau}))^*\mathbf{T} \end{array}$$

□

Lemma 1.4 (Caution)**.** *It is not necessarily the case that, supposing $\mathfrak{F}(\boldsymbol{\tau})$ is a representable map and α is a strict $\mathfrak{F}$-structure for $\boldsymbol{\tau}$, then α is cartesian.*

Proof. Suppose this were the case.

(1) Let $\mathcal{C}$ be a small category, and let $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T} : \mathcal{E}$ be a representable natural transformation, writing $\mathcal{E}$ for $\mathcal{P}r(\mathcal{C})$.
(2) Let $\mathfrak{F}_{\emptyset}$ be the constant endofunctor on $\mathcal{E}^{\rightarrow}_{cart}$ from our earlier example, sending any family to $\emptyset_{\mathcal{E}} \longrightarrow \mathbf{1}_{\mathcal{E}}$.
(3) Suppose that $\mathfrak{F}_{\emptyset}(\boldsymbol{\tau}) \xrightarrow{\alpha} \boldsymbol{\tau}$ is a strict $\mathfrak{F}$-structure for $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$.
(4) Define $\mathfrak{F}_{\mathsf{j}(\emptyset)}$ to be the constant endofunctor on $\mathcal{P}r(\mathcal{E})^{\rightarrow}_{cart}$ sending any family to $\mathsf{j}(\emptyset_{\mathcal{E}}) \longrightarrow \mathsf{j}(\mathbf{1}_{\mathcal{E}})$.
(5) $\mathsf{j}(\boldsymbol{\tau})$ is obviously a representable map in $\mathcal{P}r(\mathcal{E})$ and so is $\mathfrak{F}_{\mathsf{j}(\emptyset)}(\mathsf{j}(\boldsymbol{\tau}))$.
(6) The $\mathfrak{F}_{\emptyset}$-structure $\mathfrak{F}_{\emptyset}(\boldsymbol{\tau}) \xrightarrow{\alpha} \boldsymbol{\tau}$ lifts to a $\mathfrak{F}_{\mathsf{j}(\emptyset)}$-structure $\mathfrak{F}_{\mathsf{j}(\emptyset)}(\mathsf{j}(\boldsymbol{\tau})) \xrightarrow{\mathsf{j}(\alpha)} \mathsf{j}(\boldsymbol{\tau})$ for $\dot{\mathbf{T}} \xrightarrow{\boldsymbol{\tau}} \mathbf{T}$, using the fact that the Yoneda embedding is dense and fully faithful.
(7) Then, under our assumption, $\mathsf{j}(\alpha)$ must be cartesian.
(8) But the Yoneda embedding reflects limits, so this would imply that α is cartesian in $\mathcal{E}$, which we have already argued cannot be. □

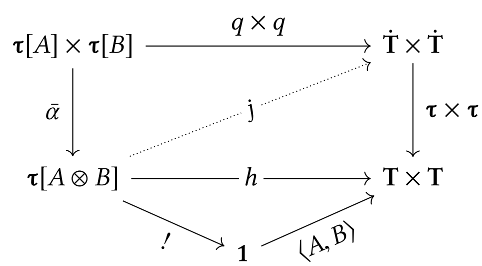
There are, however, plenty of cases where strict $\mathfrak{F}$-structures are necessarily cartesian. For instance, the $\mathfrak{F}$ that takes $\boldsymbol{\tau}$ to the "generic binary product" family for $\boldsymbol{\tau}$ will have this property.

Construction 1.5. *Let $\mathcal{E}^{\rightarrow}_{cart} \xrightarrow{\mathfrak{F}_{\times}} \mathcal{E}^{\rightarrow}_{cart}$ be the endofunctor taking each $A \xrightarrow{f} B$ to the generic binary product family $\mathfrak{F}_{\times}(f) = A^2 \xrightarrow{f^2} B^2$.*

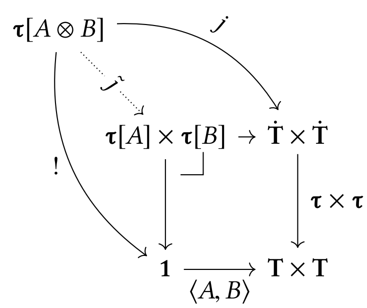
Lemma 1.6. *Let $\mathfrak{F}_{\times}(\boldsymbol{\tau}) \xrightarrow{\alpha} \boldsymbol{\tau} : \mathcal{E}^{\rightarrow}$ be a strict $\mathfrak{F}_{\times}$-structure; then α is cartesian.*

Proof. We will write $\mathbf{T}^2 \xrightarrow{\otimes} \mathbf{T}$ for $\partial_1(\alpha)$. It suffices to show that the cartesian gap map $\mathfrak{F}_{\times}(\boldsymbol{\tau}) \xrightarrow{\bar{\alpha}} \otimes^*\boldsymbol{\tau} : \mathcal{E}_{/\mathbf{T}^2}$ is an isomorphism. We will explicitly compute an inverse to $\bar{\alpha}$ using the strict $\mathfrak{F}_{\times}$-structure.

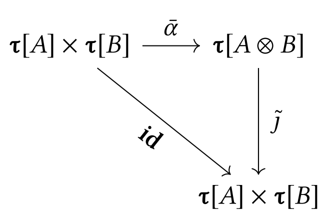
We will write A, B for the two generic elements of $(\mathbf{T}^2)^*\mathbf{T}$ in $\mathcal{E}_{/\mathbf{T}^2}$; we will also suppress the weakening re-indexings, writing $\dot{\mathbf{T}}$ for $(\mathbf{T}^2)^*\dot{\mathbf{T}}$. In this type theoretic style, it is appropriate to write $\boldsymbol{\tau}[A] \times \boldsymbol{\tau}[B]$ for $\mathfrak{F}_{\times}(\dot{\mathbf{T}})$, and $\boldsymbol{\tau}[A \otimes B]$ for $\otimes^*\boldsymbol{\tau}$. The orthogonality condition for α ensures, by transpose along the adjunction $\bullet \times 2 \dashv [\![2, \bullet]\!]$, a unique lift for the following square in $\mathcal{E}_{/\mathbf{T}^2}$:



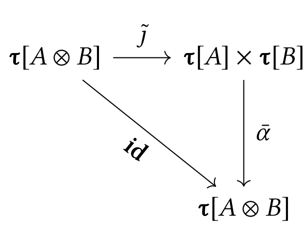
The map j induces a unique map $\boldsymbol{\tau}[A \otimes B] \xrightarrow{\tilde{j}} \boldsymbol{\tau}[A] \times \boldsymbol{\tau}[B]$



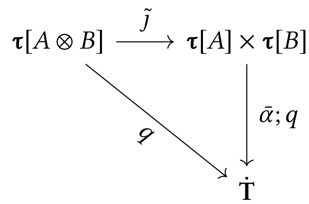
We must check that $\tilde{j}$ is a full inverse to $\bar{\alpha}$; the following triangle commutes immediately using the universal property of the product, proving that $\tilde{j}$ is a retraction of $\bar{\alpha}$:



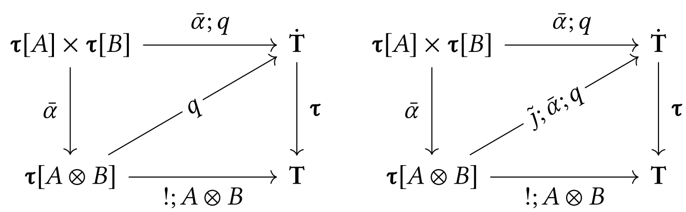
That $\tilde{j}$ is a section of $\bar{\alpha}$ will follow from the uniqueness of lifts; we are trying to check that the following triangle commutes:



Because $\boldsymbol{\tau}[A \otimes B] \overset{q}{\rightarrowtail} \dot{\mathbf{T}}$ is a monomorphism in $\mathcal{E}_{/\mathbf{T}^2}$, it suffices to check that the following triangle commutes:



By the uniqueness of lifts, it suffices to check that all the triangles below commute:



The first diagram commutes trivially, and the second commutes because $\tilde{j}$ is a retraction of $\bar{\alpha}$. ◻

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