What should a generic object be?

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Abstract

Jacobs has proposed definitions for (weak, unqualified, strong, split) generic objects for a fibered category; building on his definition of **generic object** and **split generic object**, Jacobs develops a menagerie of important fibrational structures with applications to categorical logic and computer science, including *higher order fibrations*, *polymorphic fibrations*, *λ2-fibrations*, *triposes*, and others. We observe that a split generic object need not in particular be a generic object under the given definitions, and that the definitions of polymorphic fibrations, triposes, *etc*. given by Jacobs are strict enough to rule out many fundamental examples that must be accounted for by any candidate definition. We argue for a new alignment of terminology that emphasizes the forms of generic object that appear most commonly in nature, *i.e*. in the study of internal categories, triposes, and the denotational semantics of polymorphic types. In addition, we propose a new class of acyclic generic objects inspired by recent developments in the semantics of homotopy type theory, generalizing the *realignment* property of universes to the setting of an arbitrary fibration.

Contents

1 Introduction

Since the latter half of the 20th century, *fibered category theory* or the *theory of fibrations* has played an important background role in both the applications and foundations of category theory [Gro71; Bén85]. Fibered categories, also known as fibrations, are a formalism for manipulating categories that are defined *relative* to another category, generalizing the way that ordinary categories can be thought of as being defined relative to the category of sets.

In what sense is ordinary category theory pinned to the category of sets? This can be illustrated by considering the definition of when a category $\mathscr{C}$ "has products":

Definition 1.1. A category $\mathscr{C}$ has **products** when for any indexed family $\{E_i \in \mathscr{C}\}_{i\in I}$ of objects, there exists an object $\prod_{i\in I} E_i \in \mathscr{C}$ together with a family of morphisms $p_k : \prod_{i\in I} E_i \longrightarrow E_k$ such that for any family of morphisms $h_k : H \longrightarrow E_k$ there exists a unique morphism $h : H \longrightarrow \prod_{i\in I} E_i$ factoring each h_k through p_k.

In the above, the dependency on the category **Set** is clear: the indexing object I is a set. If we had required I to be drawn from a proper subcategory of **Set** (*e.g.* finite sets) or a proper supercategory (*e.g.* classes), the notion of product defined thereby would have been different. The purpose of the formalism of *fibered categories* is to explicitly control the ambient category that parameterizes all indexed notions, such as products, sums, limits, colimits, *etc*.

Remark 1.2 (Relevance to computer science)**.** The ability to explicitly control the parameterization of products and sums is very important in theoretical computer science, especially for the denotational semantics of *polymorphic types* of the form $\forall\alpha.\tau[\alpha]$. Such a polymorphic type should be understood as the product of all $\tau[\alpha]$ indexed in the "set" of all types, but a famous result of Freyd [Fre03] shows that if a category $\mathscr{C}$ has products of this form *parameterized in* **Set**, then $\mathscr{C}$ must be a preorder. Far from bringing to an early end the study of polymorphic types in computer science, awareness of Freyd's result sparked and guided the search for ambient categories other than **Set** in which to parameterize these products [Pit87; Hyl88; HRR90]. Fibered category theory provides the optimal language to understand all such indexing scenarios, and the textbook of Jacobs [Jac99], discussed at length in the present paper, provides a detailed introduction to the applications of fibered category theory to theoretical computer science.

1.1 Introduction to fibered categories

Before giving a general definition, we will see the way that fibered categorical language indeed makes parameterization explicit by considering the *prototype* of all fibered categories, the category $\mathbf{Fam}(\mathscr{C})$ of $\mathbf{Set}$-indexed families in $\mathscr{C}$.

Construction 1.3 (The category of families)**.** We define $\mathbf{Fam}(\mathscr{C})$ to be the category of $\mathbf{Set}$-indexed families in $\mathscr{C}$, such that

1. an object of $\mathbf{Fam}(\mathscr{C})$ is a family $\{E_i \in \mathscr{C}\}_{i\in I}$ where I is a set,
2. a morphism $\{F_j\}_{j\in J} \longrightarrow \{E_i\}_{i\in I}$ in $\mathbf{Fam}(\mathscr{C})$ is given by a function $u : J \longrightarrow I$ together with for each $j \in J$ a morphism $\bar{u}_j : F_j \longrightarrow E_{uj}$.

There is an evident functor $p : \mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$ taking $(\{E_i\}_{i\in I})$ to I.

Construction 1.4 (Fiber categories)**.** For each $I \in \mathbf{Set}$, we may define the *fiber* $\mathbf{Fam}(\mathscr{C})_I$ of $\mathbf{Fam}(\mathscr{C})$ over I to be the category of I-indexed families $\{E_i \in \mathscr{C}\}_{i\in I}$ in $\mathscr{C}$, with morphisms $\{F_i\}i \in I \longrightarrow \{E_i\}i \in I$ given by morphisms $h_i : F_i \longrightarrow E_i$ for each $i \in I$.

More abstractly, the fiber category $\mathbf{Fam}(\mathscr{C})_I$ is the following pullback:

$$\begin{array}{ccc} \mathbf{Fam}(\mathscr{C})_I & \longrightarrow & \mathbf{Fam}(\mathscr{C}) \\ \big\downarrow \lrcorner & & \big\downarrow p \\ \mathbf{1} & \xrightarrow[I]{} & \mathbf{Set} \end{array}$$

Construction 1.5 (Reindexing functors)**.** For any function $u : J \longrightarrow I$, there is a corresponding reindexing functor $u^* : \mathbf{Fam}(\mathscr{C})_I \longrightarrow \mathbf{Fam}(\mathscr{C})_J$ that restricts an I-indexed family into a J-indexed family by precomposition.

With the reindexing functors in hand, can now rephrase the condition that $\mathscr{C}$ has products (Definition 1.1) in terms of $\mathbf{Fam}(\mathscr{C})$.

Proposition 1.6. *A category $\mathscr{C}$ has products if and only if for each product projection function $\pi_{I,J} : I \times J \longrightarrow I$, the reindexing functor $\pi_{I,J} : \mathbf{Fam}(\mathscr{C})_I \longrightarrow \mathbf{Fam}(\mathscr{C})_{I\times J}$ has a right adjoint $\prod_{(I,J)} : \mathbf{Fam}(\mathscr{C})_{I\times J} \longrightarrow \mathbf{Fam}(\mathscr{C})_I$ such that the following* Beck–Chevalley *condition holds: for any function $u : K \longrightarrow I$, the canonical natural transformation $u^* \circ \prod_{(I,J)} \longrightarrow \prod_{(K,J)} \circ (u \times \mathsf{id}_J)^*$ is an isomorphism.*

The characterization of products in terms of the category of families may seem more complicated, but it has a remarkable advantage: we can replace $\mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$ with a different functor satisfying similar properties in order to speak more generally of when one category has "products" that are parameterized in another category. The properties that this functor has to satisfy for the notion to make sense are embodied in the definition of a *fibration* or *fibered category*; a functor $\mathscr{E} \longrightarrow \mathscr{B}$ will be called a fibration when it behaves similarly to the functor projecting the parameterizing object from a category of families of objects. We begin with an auxiliary definition of *cartesian morphism*:

Definition 1.7. Let $p : \mathscr{E} \longrightarrow \mathscr{B}$ and let $E \longrightarrow F$ be a morphism in $\mathscr{E}$, which we depict as follows:

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ pE & \longrightarrow & pF \end{array}$$

In the diagram above, we say that $E \longrightarrow F$ **lies over** $pE \longrightarrow pF$. We say that $E \longrightarrow F$ is **cartesian** in p when for any morphism $H \longrightarrow F$ in $\mathscr{E}$ and $pH \longrightarrow pE$ in $\mathscr{B}$, there exists a unique morphism $H \longrightarrow E$ lying over $pH \longrightarrow pE$ such that $H \longrightarrow F$ lies over the composite $pH \longrightarrow pE$ as depicted:

$$\begin{array}{ccccc} & & \overset{}{\frown\!\!\longrightarrow} & & \\ H & \overset{\exists!}{\dashrightarrow} & E & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ pH & \longrightarrow & pE & \longrightarrow & pF \end{array}$$

Remark 1.8 (Explication of cartesian maps)**.** Returning to our example of the category of families $\mathbf{Fam}(\mathscr{C})$ over $\mathbf{Set}$, we can make sense of the notion of a cartesian map. Given a function $u : J \longrightarrow I$ of indexing sets, the reindexing functor u^* takes I-indexed families to J-indexed families. Given an I-indexed family $\{E_i\}_{i\in I}$, we may define a morphism $u^*\{E_i\}_{i\in I} \longrightarrow \{E_i\}_{i\in I}$ in $\mathscr{C}$ whose first component is $u : J \longrightarrow I$ and whose second component is the identity function $E_{uj} \longrightarrow E_{uj}$ at each $j \in J$. The morphism $u^*\{E_i\}_{i\in I} \longrightarrow \{E_i\}_{i\in I}$ is then *cartesian* in $p : \mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$.

Exercise 1.9. Verify that the morphism $u^*\{E_i\}_{i\in I} \longrightarrow \{E_i\}_{i\in I}$ constructed in Remark 1.8 is indeed cartesian in $p : \mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$.

There is another way to understand cartesian maps, suggested by the name.

Exercise 1.10. Let $\mathscr{B}^{\rightarrow}$ be the *arrow category* of $\mathscr{B}$, whose objects are morphisms of $\mathscr{B}$ and whose morphisms are commuting squares between them; let $\mathsf{cod} : \mathscr{B}^{\rightarrow} \longrightarrow \mathscr{B}$ be the *codomain functor* that projects the codomain of a map $A \longrightarrow B$. Show that a morphism $E \longrightarrow F \in \mathscr{B}^{\rightarrow}$ is cartesian if and only if the corresponding square in $\mathscr{B}$ is a pullback square (also called a cartesian square).

Finally we may give the definition of a fibration.

Definition 1.11. A functor $p : \mathscr{E} \longrightarrow \mathscr{B}$ is called a **fibration** when for any object $E \in \mathscr{E}$ and morphism $B \longrightarrow pE \in \mathscr{B}$ there exists a cartesian morphism $H \longrightarrow E$ lying over $B \longrightarrow pE$. The cartesian morphism is often called the **cartesian lift** of $B \longrightarrow pE$.

Convention 1.12. We will depict a fibration $p : \mathscr{E} \longrightarrow\!\!\triangleright\, \mathscr{B}$ using triangular arrows. When we wish to leave the functor implicit, we refer to $\mathscr{E}$ as a **fibered category** over $\mathscr{B}$. In the same way that one writes $A \times B$ for the apex of a product diagram, we will often write $\bar{u} : u^*E \longrightarrow E$ for the cartesian lift of $u : B \longrightarrow pE$ as depicted below:

$$
\begin{array}{ccc}
u^*E & \xrightarrow{\bar{u}} & E \\
\downarrow & & \downarrow \\
B & \xrightarrow[u]{} & pE
\end{array}
$$

In the case of $\mathbf{Fam}(\mathscr{C})$, the existence of cartesian lifts for each $u : B \longrightarrow pE$ corresponds to the *reindexing* functors $u^* : \mathbf{Fam}(\mathscr{C})_{pE} \longrightarrow \mathbf{Fam}(\mathscr{C})_B$.

Exercise 1.13. Verify that the functor $\mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$ is a fibration.

Exercise 1.14. Conclude from Exercise 1.10 that the codomain functor $\mathscr{B}^{\to} \longrightarrow \mathscr{B}$ is a fibration if and only if $\mathscr{B}$ has all pullbacks.

When the codomain functor $\mathscr{B}^{\to} \longrightarrow \mathscr{B}$ is a fibration, we will refer to it as the **fundamental fibration**, written $\mathbf{P}_{\mathscr{B}}$ following Streicher [Str21].

1.2 Small categories and generic objects

An ordinary category need not have a set of objects — for instance, the category **Set** of all sets has a *proper class* of objects. Likewise, it is possible to find categories such that between two objects there may be a proper class of morphisms (*e.g.* the category of spans). A category that has hom *sets* is called **locally small**, and a category that has a *set* of objects is called **globally small**. A category that has both is just called **small**. Small categories are very useful: for instance, if $\mathbb{C}$ is a small category then the category of functors $[\mathbb{C}, \mathbf{Set}]$ is a Grothendieck topos. Functor categories of this kind play an important role in theoretical computer science [*e.g.* Ole86; Rey95; Bir+11].

The idea of a (globally, locally) small category can be relativized from **Set** to another category in two *a priori* different ways that ultimately coincide up to equivalence. The simplest and move naïve way to think of a small category $\mathbb{C}$ in a category $\mathscr{B}$ is as an *algebra* for the sorts and operations of the *theory of a category* internal to $\mathscr{B}$.

Definition 1.15. Let $\mathscr{B}$ be a category that has pullbacks. An **internal category** or **category object** in $\mathscr{B}$ is given by:

1. an object $\mathbb{C}_0 \in \mathscr{B}$ of *objects*,
2. and an object $\mathbb{C}_1 \in \mathscr{B}$ of *morphisms*,
3. and source and target maps $s, t : \mathbb{C}_1 \longrightarrow \mathbb{C}_0$,
4. and a morphism $i : \mathbb{C}_0 \longrightarrow \mathbb{C}_1$ choosing the identity maps, such that $s \circ i = \mathsf{id}_{\mathbb{C}_0} = t \circ i$,
5. and a morphism $c : \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \longrightarrow \mathbb{C}_1$ choosing composite maps such that $s \circ c = s \circ \pi_1$ and $t \circ c = t \circ \pi_2$,

6. (and several other equations associativity and unit laws of composition)

Observation 1.16. *When* $\mathscr{B} = \mathbf{Set}$ *we obtain exactly the ordinary notion of a small category,* i.e. *a small category is the same thing as an* internal category *or* category object *in* **Set**.

In the previous section, we have argued that fibrations are a fruitful way to think about categories defined relative to another category. Indeed, we may view an internal category $\mathbb{C}$ as a fibration via a process called *externalization*. This proceeds in two steps; first we construct a *presheaf of categories* on $\mathscr{B}$, and then we use the *Grothendieck construction* to turn it into a fibration.

Construction 1.17 (The presheaf of categories associated to an internal category)**.** Let $\mathbb{C}$ be an internal category in $\mathscr{B}$. We may define a presheaf of categories $\mathbb{C}^\bullet : \mathscr{B}^\circ \longrightarrow \mathbf{Cat}$ like so:

1. for $I \in \mathscr{B}$, an object of $\mathbb{C}^I$ is given by a morphism $\alpha : I \longrightarrow \mathbb{C}_0$,
2. for $I \in \mathscr{B}$, a morphism $\alpha \longrightarrow \beta \in \mathbb{C}^I$ is given by a morphism $h : I \longrightarrow \mathbb{C}_1$ such that $s \circ h = \alpha$ and $t \circ h = \beta$,
3. for $u : J \longrightarrow I$ in $\mathscr{B}$, the reindexing $u^* : \mathbb{C}^I \longrightarrow \mathbb{C}^J$ is given on both objects and morphisms by precomposition with u.

Observe that the presheaf of categories associated to an internal category is, in each fiber I, the category object in **Set** obtained by restricting along the functor $I : \mathbf{1} \longrightarrow \mathscr{B}$.

Construction 1.18 (The Grothendieck construction)**.** Let $\mathbb{C}^\bullet : \mathscr{B}^\circ \longrightarrow \mathbf{Cat}$ be a presheaf of categories; we define its *total category* $\int_{\mathscr{B}} \mathbb{C}^\bullet$ as follows:

1. an object of $\int_{\mathscr{B}} \mathbb{C}^\bullet$ is given by a pair of an object $I \in \mathscr{B}$ and an object $c \in \mathbb{C}^I$,
2. a morphism $(J, c) \longrightarrow (I, d)$ is given by a pair of a morphism $u : J \longrightarrow I \in \mathscr{B}$ and a morphism $c \longrightarrow \mathbb{C}^u d$ in $\mathbb{C}^J$.

There is an evident functor $p : \int_{\mathscr{B}} \mathbb{C}^\bullet \longrightarrow \mathscr{B}$; it is this functor that is referred to as the *Grothendieck construction*.

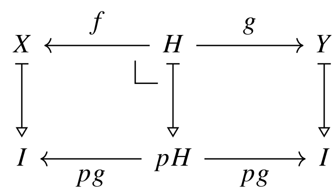
Exercise 1.19. Verify that the Grothendieck construction of any presheaf of categories $\mathbb{C}^\bullet : \mathscr{B}^\circ \longrightarrow \mathbf{Cat}$ is a fibration.

Definition 1.20 (Externalization of an internal category)**.** Let $\mathbb{C}$ be an internal category in $\mathscr{B}$; its **externalization** is defined to be the Grothendieck construction $[\mathbb{C}] := \int_{\mathscr{B}} \mathbb{C}^\bullet \longrightarrow \mathscr{B}$ of the associated presheaf of categories (Construction 1.17).

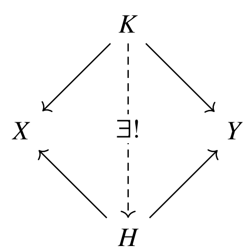
As promised we may now isolate the properties of the fibered category $[\mathbb{C}]$ that correspond (up to equivalence) to arising by externalization from an internal category.

Definition 1.21. A fibered category $\mathscr{E} \longrightarrow \mathscr{B}$ is called **globally small**[1] if there is an object $T \in \mathscr{E}$ such that for any $X \in \mathscr{E}$ there exists a cartesian map $X \longrightarrow T$.

Definition 1.22. A fibered category $\mathscr{E} \longrightarrow \mathscr{B}$ is called **locally small** when for any $I \in \mathscr{B}$ and $X, Y \in \mathscr{E}_I$ there exists a span $X \leftarrow H \rightarrow Y$ in the following configuration,



such that for any other span $X \leftarrow K \rightarrow Y$ so-configured, there is a unique map $K \longrightarrow H$ making the the following diagram commute:



Exercise 1.23 (Difficult)**.** Let $\mathscr{C}$ be an ordinary category; show that the fibered category $\mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$ is locally small if and only if $\mathscr{C}$ is locally small. Show that $\mathscr{C}$ is equivalent to a small category if and only if $\mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$ is both globally small and locally small.

One of the fundamental results of fibered category theory is that, up to equivalence, global and local smallness in the sense of Definitions 1.21 and 1.22 suffice to detect internal categories.

Proposition 1.24. *A fibration $\mathscr{E} \longrightarrow \mathscr{B}$ is equivalent to the externalization of an internal category $\mathbb{E}$ if and only if it is both globally and locally small.*

Although we do not include the (standard) proof of Proposition 1.24, it is instructive to understand the object $T \in [\mathbb{E}]$ in the externalization of an internal category $\mathbb{E}$ that renders $[\mathbb{E}]$ globally small. Recalling the definition of the externalization via the Grothendieck construction, we define T to be the pair $\left(\mathbb{C}_0, \mathsf{id}_{\mathbb{C}_0} : \mathbb{C}_0 \longrightarrow \mathbb{C}_0\right)$ given by the object of objects and its identity map. Viewed from the internal perspective, $\mathsf{id}_{\mathbb{C}_0}$ is the generic object of $\mathbb{C}$ in the slice $\mathscr{B} \downarrow \mathbb{C}_0$.

[1] Note that our usage here differs from that of Jacobs [Jac99] in a critical way. In fact, the bulk of the present paper is devoted to explaining *why* it is not optional to deviate from the usage of *op. cit*.

1.3 Goals and structure of this paper

Although the property stated in Definition 1.21, that there exist cartesian morphisms $X \longrightarrow T$ for any X, is the most that can be required for an arbitrary internal category, more restrictive notions of generic object have proved important in practice for different applications. Unfortunately, over the years a number of competing definitions have proliferated throughout the literature — and some of the more established of these definitions lead to false conclusions when taken too literally, as we point out in Section 3.1 in our discussion of Jacobs' mistaken Corollary 9.5.6.

The goal of this paper, therefore, is to argue for a new alignment of terminology for the different forms of generic object that is both internally consistent *and* reflects the use of generic objects in practice. Because generic objects play an important role in several areas of application (categorical logic, algebraic set theory, homotopy type theory, denotational semantics of polymorphism, *etc*.), we believe that we have sufficient evidence today to correctly draw the map.

- In Section 2, we recall the definitions of several variants of generic object by Jacobs [Jac99]; our main observation is that a **split generic object** in the sense of *op. cit*. need not be a **generic object** in the same sense.
- In Section 3, we analyze the consequences of the definitions discussed in Section 2 for the use of generic objects in internal category theory, tripos theory, denotational semantics of polymorphism, algebraic set theory, and homotopy type theory.
- In Section 4, we propose new unified terminology and definitions for all extant forms of generic object (as well as one *new* one). Our proposal is summarized and compared with the literature in Table 1.

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2 Four kinds of generic object

We begin by recalling Definition 5.2.8 of Jacobs [Jac99], from which we some additional characterizations that will not play a role in our analysis.

> *Consider a fibration $p : \mathscr{E} \longrightarrow\!\!\triangleright\, \mathscr{B}$ and an object T in the total category $\mathscr{E}$. We call T a*
>
> *i)* ***weak generic object*** *if*
>
> $$\forall X \in \mathscr{E}. \exists f : X \longrightarrow T. f \textit{ is cartesian}$$

ii) ***generic object*** *if*

$$\forall X \in \mathscr{E}.\exists! u : pX \longrightarrow pT.\exists f : X \longrightarrow T. f \text{ is cartesian over } u$$

iii) ***strong generic object*** *if*

$$\forall X \in \mathscr{E}.\exists! f : X \longrightarrow T. f \text{ is cartesian}$$

Jacobs [Jac99] then defines **split generic objects** for split fibrations in Definition 5.2.1, paraphrased below:

> A split fibration $p : \mathscr{E} \longrightarrow\!\!\triangleright\, \mathscr{B}$ has a **split generic object** if there is an object $\Omega \in \mathscr{B}$ together with natural isomorphism $\theta : \mathscr{B}(-,\Omega) \longrightarrow \mathsf{ob}\,\mathscr{E}_\bullet$ in $[\mathscr{B}^\circ, \mathbf{Set}]$, where the presheaf $\mathsf{ob}\,\mathscr{E}_\bullet$ is defined using the splitting.

A useful characterization of **split generic objects** is given in Lemma 5.2.2 of *op. cit.*:

> *A split fibration $p : \mathscr{E} \longrightarrow\!\!\triangleright\, \mathscr{B}$ has a* ***split generic object*** *if and only if there is an object $T \in \mathscr{E}$ with the property that $\forall X \in \mathscr{E}.\exists! u : pX \longrightarrow pT. u^*T = X$. [Jac99]*

Scholium 2.1. The **weak** and **strong generic objects** of Jacobs [Jac99] are referred to by Phoa [Pho92] as *generic objects* and *skeletal generic objects*. Phoa [Pho92] does not consider the intermediate notion. On the other hand, Phoa [Pho92] defines *strict generic objects* relative to an arbitrary (non-split) cleavage; a **split generic object** is indeed a *strict generic object* in the sense of Phoa, but even for a split cleavage, a *strict generic object* need not be a **split generic object**. We will discuss Phoa's terminology more in Scholium 4.4.

2.1 Separating generic objects from strong generic objects

Jacobs [Jac99] notes that generic and strong generic objects coincide in preordered fibrations, but they may differ otherwise — the difference emanating from the presence of non-trivial vertical automorphisms. We provide a concrete example that separates generic and strong generic objects:

Example 2.2 (A generic object that is not strong generic)**.** Let W be the set of ordinals in a universe U; using the axiom of choice, there is a map $s : W \longrightarrow U$ picking out a set representing each ordinal. Let $\bar{W}$ be the set of pairs of an ordinal α together with an element of a $x \in s\alpha$. The forgetful map $\bar{W} \longrightarrow W$ determines a full internal subcategory $\mathbb{W}$ of **Set**; we note that $\bar{W} \in [\mathbb{W}]_W$ is a **generic object** for $[\mathbb{W}]$.

Suppose however that $\bar{W} \in [\mathbb{W}]_W$ is a ***strong*** **generic object**, and consider the ordinal $\mathbf{2} \in [\mathbb{W}]_{\{*\}}$ over the point; $s\mathbf{2}$ is an arbitrary two-element set, which we will write $\{0, 1\}$. Then there exists a unique cartesian morphism $\mathbf{2} \longrightarrow \bar{W}$; but we have the following two cartesian squares:

$$
\begin{array}{ccc}
\mathbf{2} & \xrightarrow{\{0 \mapsto (\mathbf{2}, 0),\, 1 \mapsto (\mathbf{2}, 1)\}} & \bar{W} \\
\downarrow & \lrcorner & \downarrow \\
\{*\} & \xrightarrow[\mathbf{2}]{} & W
\end{array}
$$

$$
\begin{array}{ccc}
\mathbf{2} & \xrightarrow{\{0 \mapsto (\mathbf{2}, 1),\, 1 \mapsto (\mathbf{2}, 0)\}} & \bar{W} \\
\downarrow & \lrcorner & \downarrow \\
\{*\} & \xrightarrow[\mathbf{2}]{} & W
\end{array}
$$

By assumption, the upper maps are unique and thus equal (a contradiction).

In fact, the generic object of Example 2.2 can moreover be seen to be split. Thus our observations contradict the following claim of Hermida [Her93, Examples 1.4.12]:

> *For a category $\mathbb{C}$, the family fibration* **Fam**$(\mathbb{C})$ *has a generic object precisely when $\mathbb{C}$ has a small set of objects. In this case, the* ***strong generic object*** *is $\{x\}_{x \in \mathbb{C}_0}$. [Her93]*

Indeed, it is not possible for $\{x\}_{x \in \mathbb{C}_0}$ to be a **strong generic object** when $\mathbb{C}$ has non-trivial automorphisms. It seems likely that the word "strong" was added inadvertently, as Hermida uses the term *generic object* to refer to Jacobs' **weak generic object**.

Example 2.2 is the prototype for an entire class of (quite artificial) examples of **generic objects** constructed in the externalizations of internal categories using the axiom of choice, which we will discuss in Section 2.3. Before we get there, we will discuss externalizations of internal categories and their (weak, split) generic objects.

2.2 A split generic object need not be a generic object

Definition 2.3 (Externalization of an internal category)**.** Let $\mathscr{B}$ be a category with finite limits, and let $\mathbb{C}$ be an internal category in $\mathscr{B}$. The externalization $[\mathbb{C}]$ is a fibered category over $\mathscr{B}$:

1. for $U \in \mathscr{B}$, a displayed object in $[\mathbb{C}]_U$ is a morphism $c : U \longrightarrow \mathbb{C}_0$;

2. for $f : U \longrightarrow V \in \mathscr{B}$ and $c \in [\mathbb{C}]_U$ and $d \in [\mathbb{C}]_V$, a displayed morphism over f is a morphism $\bar{f} : U \longrightarrow \mathbb{C}_1$ such that $\bar{f}; s = c$ and $\bar{f}; t = f; d$.

Thus the total category $[\mathbb{C}]$ is the category of pairs (U, c) as above, *etc*.

Construction 2.4 (The canonical splitting of the externalization)**.** The externalization of an internal category is split in a canonical way: given $(I, c) \in [\mathbb{C}]$ and $u : J \longrightarrow I$, we choose $u^*(I, c) = (J, c \circ u)$. The cartesian morphism $u^*(I, c) \longrightarrow (I, c)$ is given by the pair $(u, \mathsf{id} \circ c \circ u)$ where $\mathsf{id} : \mathbb{C}_0 \longrightarrow \mathbb{C}_1$ is the generic identity morphism.

Construction 2.5 (Weak, split generic objects in the externalization)**.** The externalization $p : [\mathbb{C}] \longrightarrow \mathscr{B}$ has a **weak generic object** $T = (\mathbb{C}_0, \mathsf{id}_{\mathbb{C}_0})$. Relative to the splitting of $[\mathbb{C}]$ from Construction 2.4, the **weak generic object** T is also a **split generic object**.

The **weak generic object** of the externalization of an internal category defined in Construction 2.5 obviously need not be a **strong generic object**, but it may be more surprising to learn that it also need not be a **generic object** at all. This can happen, for instance, when the internal category $\mathbb{C}$ has two distinct isomorphic objects; the following concrete example illustrates the problem:

Example 2.6 (A split generic object that is not a generic object)**.** Letting U be a Grothendieck universe in **Set**, we may consider the *full internal subcategory* $\mathbb{U}$ determined by U. The externalization $[\mathbb{U}]$ is a fibered category over **Set** and comes equipped with a canonical splitting defined as on Construction 2.4; relative to this splitting, $\bar{U} = [A \mid A \in U] \in [\mathbb{U}]_U$ is a split generic object for $[\mathbb{U}]$. It is not, however, a generic object. For the counterexample, assume that $\bar{U} \in [\mathbb{U}]_U$ is a generic object and let $\{0, 1\} \in [\mathbb{U}]_{\{*\}}$ be a two-element set over the point. Then there exists a unique function $b : \{*\} \longrightarrow U$ such that there exists a cartesian morphism $\bar{b} : \{0, 1\} \longrightarrow \bar{U}$ lying over b in the following sense, by definition. However, we have the following two cartesian squares:

$$\begin{array}{ccc} \{0,1\} & \xrightarrow{\{0 \mapsto (\{0,1\},0),\, 1 \mapsto (\{0,1\},1)\}} & \bar{U} \\ \downarrow & \lrcorner & \downarrow \\ \{*\} & \xrightarrow[\{0,1\}]{} & U \end{array}$$

$$\begin{array}{ccc} \{0,1\} & \xrightarrow{\{0 \mapsto (\{2,3\},2),\, 1 \mapsto (\{2,3\},3)\}} & \bar{U} \\ \downarrow & \lrcorner & \downarrow \\ \{*\} & \xrightarrow[\{2,3\}]{} & U \end{array}$$

By uniqueness of the lower map with the property of lying underneath a cartesian morphism $\{0, 1\} \longrightarrow \bar{U}$, we conclude that $\{0, 1\} = \{2, 3\} \in U$, a contradiction.

Corollary 2.7. *A split generic object is not necessarily a generic object.*

2.3 Generic objects from weak generic objects

We have seen that the externalization of an internal category $\mathbb{C}_0$ in $\mathscr{B}$ has an obvious **weak generic object** that is also split; the **weak generic object** $T \in [\mathbb{C}]$ is simply the identity map $\mathsf{id}_{\mathbb{C}_0} : \mathbb{C}_0 \longrightarrow \mathbb{C}_0$ in $\mathscr{B}$. Nonetheless, in some cases the externalization of an internal category may have a **generic object** T', usually different from the **weak generic object** T described above.

Construction 2.8 (Quotienting the weak generic object of an externalization)**.** Suppose that $\mathscr{B} = \mathbf{Set}$ and thus $\mathbb{C}$ is an ordinary small category. Then we may consider the quotient $\mathbb{C}_0/\cong$ of the objects of $\mathbb{C}$ under isomorphism; in other words, this is the set of isomorphism classes of $\mathbb{C}$-objects. Using the axiom of choice, we may arbitrarily choose a section $s : \mathbb{C}_0/\cong \longrightarrow \mathbb{C}_0$ to the quotient map; moreover, we may choose a function associating to each $u \in \mathbb{C}_0$ an isomorphism $u \cong s[u]_{/\cong}$.

Lemma 2.9. *The pair $T' = (\mathbb{C}_0/\cong, s)$ is a **generic object** for* $[\mathbb{C}]$.

Proof. Fixing $(I, c) \in [\mathbb{C}]$ we must choose a unique $u : I \longrightarrow pT'$ such that there exists a cartesian map $(I, c) \longrightarrow T$ lying over u. We choose $u(i) = [c(i)]_{/\cong}$, taking each index $i \in I$ to the isomorphism class of $c(i)$.

1. First of all, it is clear that there exists a cartesian map lying over u in the correct configuration.
2. Fixing $v : I \longrightarrow pT'$ such that there exists a cartesian map $(I, c) \longrightarrow T$ lying over v, it remains to show that $v = u$. This follows because such a cartesian map ensures that v and u are the same family of isomorphism classes of objects. □

Lemma 2.10. *If the T' defined above is a **split generic object**, then $\mathbb{C}$ is skeletal.*

Proof. We have already seen that T is a **split generic object**; but **split generic objects** are unique up to unique isomorphism, hence if T' is split then we must have an isomorphism $T \cong T'$. □

The following result is an abstraction of Example 2.2.

Lemma 2.11. *If the T' defined above is a **strong generic object**, then $\mathbb{C}$ is* gaunt *in the sense that any automorphism is an identity map.* [2]

Proof. Let $f : c \longrightarrow c$ be an automorphism in $\mathbb{C}$, *i.e.* a vertical isomorphism in $[\mathbb{C}]_{\{*\}}$. Since T' is **strong generic**, there exists a unique cartesian morphism $(\{*\}, c) \longrightarrow T'$; this means that there is a unique element of $[c] \in \mathbb{C}_0/\cong$ and a unique isomorphism $h : c \longrightarrow s[c]$. Writing $\phi_c : c \cong s[c]$ for the (globally) chosen isomorphism, we have $f; \phi_c = h = \phi_c$ and hence $f = \mathsf{id}_c$, so $\mathbb{C}$ is gaunt. □

Thus we conclude that although the family fibration $[\mathbb{C}]$ over **Set** of a small category $\mathbb{C}$ does have a generic object, this generic object cannot be either a **split generic object** or a **strong generic object** except in somewhat contrived scenarios.

[2] The *gaunt* terminology was introduced by Barwick and Schommer-Pries [BS11]; see also the nLab [nLa21]. Johnstone [Joh02] has spoken of *stiff categories* to refer to the same thing.

2.4 Weak generic objects are the correct generalization of split generic objects

It is clear that any split generic object is in particular a weak generic object; but the converse *also* holds in a certain sense that we make precise below.

Construction 2.12 (A presheaf of categories)**.** Let $p : \mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ be a fibered category and let $T \in \mathscr{E}$ be a **weak generic object** for T. We may construct a presheaf of categories $\mathscr{E}^\bullet : \mathscr{B}^\circ \longrightarrow \mathbf{Cat}$ like so:

1. an object of $\mathscr{E}^I$ is a morphism $\alpha : I \longrightarrow T$,
2. a morphism $\alpha \longrightarrow \beta$ in $\mathscr{E}^I$ is a given by
   (a) a cartesian map $\bar{\alpha} : A \longrightarrow T$ over α,
   (b) a cartesian map $\bar{\beta} : B \longrightarrow T$ over β,
   (c) a vertical map $h : A \longrightarrow B$ over I,

   such that $(\bar{\alpha}, \bar{\beta}, h)$ is identified with $(\bar{\alpha}', \bar{\beta}', h')$ when h and h' are equal modulo the unique vertical isomorphisms between the cartesian lifts A, B.

Construction 2.13. The Grothendieck construction $q : \int \mathscr{E}^\bullet \longrightarrow\!\!\triangleright \mathscr{B}$ has a canonical splitting. Given $(J, \alpha) \in \int \mathscr{E}^\bullet$ and $u : I \longrightarrow J$,

1. the object $u^*(J, \alpha)$ is chosen to be $(I, \alpha \circ u)$,
2. the cartesian morphism $u^*E \longrightarrow E$ over u in $\int \mathscr{E}^\bullet$ is defined to be the pair $(u, (\bar{\alpha}, \bar{\alpha}, \mathsf{id}_A))$ where $\bar{\alpha} : A \longrightarrow T$ is an arbitrary cartesian map over $\alpha \circ u : J \longrightarrow pT$.

This is well-defined via the axiom of choice because of the quotienting.

Lemma 2.14. *There is a fibered equivalence from* $\int \mathscr{E}^\bullet$ *to* $\mathscr{E}$ *over* $\mathscr{B}$.

Lemma 2.15. *The pair* $T' = (T, \mathsf{id}_T)$ *is a* ***split generic object*** *in* $\int \mathscr{E}^\bullet \longrightarrow\!\!\triangleright \mathscr{B}$.

Proof. Fixing $X \in \int \mathscr{E}^\bullet$, we must check that there exists a unique $u : pX \longrightarrow pT'$ such that $u^*T' = X$. Unfolding definitions, we fix $I \in \mathscr{B}$ and $\alpha : I \longrightarrow T$ to check that there is a unique $u : I \longrightarrow T$ such that $(I, u) = (I, \alpha)$. Of course, this is true with $u = \alpha$. □

Thus a **weak generic object** for a fibration $\mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ generates in a canonical way a new *equivalent* **split** fibration $\mathscr{E}' \longrightarrow\!\!\triangleright \mathscr{B}$ that has a **split generic object**. This is the correspondence between **weak generic objects** and **split generic objects**.

3 Consequences for internal categories, tripos theory, polymorphism, *etc*.

We recall several definitions from Jacobs [Jac99] below:

1. *A **higher order fibration** is a first order fibration with a **generic object** and a cartesian closed base category. Such a higher order fibration will be called **split** if the fibration is split and all of its fibred structure (including the **generic object**) is split. [Jac99, Definition 5.3.1]*

2. *A **tripos** is a higher order fibration $\mathscr{E} \longrightarrow$ **Set** over **Set** for which the induced products $\prod_u$ and coproducts $\coprod_u$ along an arbitrary function u satisfy the Beck-Chevalley condition. [Jac99, Definition 5.3.3]*

3. *A **polymorphic fibration** is a fibration with a generic object, with fibred finite products and with finite products in its base category. It will be called **split** whenever all this structure is split. [Jac99, Definition 8.4.1].*

4. *A **polymorphic fibration** with Ω in the base as a generic object will be called*

   (a) a $\lambda\!\rightarrow$-fibration if it has fibred exponents;

   (b) a $\lambda 2$-fibration if it has fibred exponents and also simple Ω-products and coproducts;

   (c) a $\lambda\omega$-fibration if it has fibred exponents, simple products and coproducts, and exponents in its base category. [Jac99, Definition 8.4.3]

5. *Let $\diamond$ be $\rightarrow, 2, \omega$. A $\lambda\diamond$-fibration will be called **split** if all of its structure is split. In particular, its underlying polymorphic fibration is split. [Jac99, Definition 8.4.4]*

A **split generic object** need not be a **generic object** as we have seen in Section 2.2, and indeed, is only quite rarely a generic object. Consequently, a **split polymorphic fibration** is not the same thing as a split fibration with a **split generic object** and split fibred finite products. This disorder suggests that a change of definitions is in order, which we propose in Section 4.

3.1 Consequences for internal category theory

The somewhat chaotic status of **generic objects** *vis-à-vis* **split generic objects** has led to an erroneous claim by Jacobs [Jac99] that the externalization of an internal category has a **generic object**. What is actually proved by *op. cit*. in Lemma 7.3.2 is that the externalization of an internal category has a **split generic object**, but this is later claimed (erroneously) to give rise to a **generic object** in the proof of Corollary 9.5.6. Thus the claimed result of Corollary 9.5.6, that a fibration is small if and only if it is globally small (has a **generic object**) and locally small, is mistaken. We suppose that if this problem had been noticed, the definition of **globally small category** by *op. cit*. would have assumed instead a **weak generic object**.

3.2 Consequences for tripos theory

Jacobs [Jac99] sketches in Example 5.3.4 the standard construction of a tripos from a partial combinatory algebra, referring to Hyland, Johnstone, and Pitts [HJP80] for several parts of the construction (including the **generic object**); it is true that this construction does indeed give rise to a **tripos** in the sense of Jacobs [Jac99], but nonetheless the standard definition of a tripos involves a **weak generic object** only [HJP80; Pit81; Pit02], a discrepancy that has already been noted by Birkedal [Bir00, p. 110].

3.3 Consequences for polymorphism

The reception in the community studying polymorphism has been to either avoid or tacitly correct the definition of generic object. For instance, Hermida [Her93] speaks of **weak generic objects** and **strong generic objects**, and gives the correct definition of $\lambda\!\rightarrow, \lambda 2, \lambda\omega$ fibration in terms of **weak generic objects**. On the other hand Birkedal, Møgelberg, and Petersen [BMP05], Johann and Sojakova [JS17], Sojakova and Johann [SJ18], and Ghani, Forsberg, and Orsanigo [GFO19] deal mainly with the **split generic objects**, and thus do not seem to run into problems. It can be seen, however, that the examples of (split) $\lambda\diamond$-fibrations in the cited works are *not* in fact (split) $\lambda\diamond$-fibrations in the sense of Jacobs [Jac99], because they do not have **generic objects**. Of course, the problem lies with the definitions rather than the examples.

3.4 Consequences for type theory and algebraic set theory

The idea of a *universe* in a category has been abstracted from Grothendieck's universes [AGV72] by way of the contributions of a number of authors including Bénabou [Bén73], Martin-Löf [Mar75], Joyal and Moerdijk [JM95], Hofmann and Streicher [HS97], and Streicher [Str05]. In fibered categorical language, a universe usually is a *full internal subcategory* of some ambient category and thus has a **weak generic object**.

In certain contrived cases, the **weak generic object** of a universe is also a **generic object** that is nonetheless not **strong**; one example is Voevodsky's construction of a universe of well-ordered simplicial sets in the context of the simplicial model of homotopy type theory, reported by Kapulkin and Lumsdaine [KL21], which is a somewhat more sophisticated version of our Example 2.2. On the other hand, universes of propositions (*e.g.* the subobject classifier of a topos or the strong subobject classifier of a quasitopos) are the main source of **strong generic objects** in nature.

The theory of universes as applied to *type theory* on the one hand and *algebraic set theory* on the other hand motivates two additional variants of generic object:

1. In applications to type theory, it has been increasingly important for universes to have a **weak generic object** T that satisfies an additional *realignment* property relative to a class of monomorphisms $\mathcal{M}$; in particular, given a cartesian span $U \leftarrowtail X \rightarrow T$ where $p(U \leftarrowtail X) \in \mathcal{M}$, we need a strict extension $U \rightarrow T$ factoring $X \rightarrow T$ through $U \rightarrowtail X$. This realignment property has proved essential for the semantics of univalent universes in homotopy type theory [KL21; Shu15; Shu19; GSS22] as well as Sterling's *synthetic Tait computability* [Ste21], an abstraction

Our Proposal	**Jacobs [Jac99]**	**Phoa [Pho92]**	**Hermida [Her93]**	**Streicher [Str05]**
weak generic	—	—	—	weak generic
generic	weak generic	generic	generic	generic
acyclic generic	—	—	—	—
skeletal generic	generic	—	—	—
gaunt generic	strong generic	skeletal generic	strong generic	classifying

Table 1: A Rosetta stone for generic objects, with different notions given in order of increasing strength.

of Artin gluing that has been used to prove several metatheoretic results in type theory and programming languages [SH21; SA21; Gra21; GB21; Niu+22; SH22].

In Section 4.3 we will discuss the generalization of the realignment property to an arbitrary fibered category as the **acyclic generic object**, which lies strictly between **weak generic objects** and **generic objects**.

2. In algebraic set theory, universes were considered that enjoy a form of generic object that is even weaker than **weak generic object**. For each $E \in \mathscr{E}$ one only "locally" has a morphism into T; the "very weak" generic objects of algebraic set theory seem to relate to **weak generic objects** in the same way that weak completeness relates to strong completeness in the context of stack completions [Hyl88; HRR90], with important applications to the theory of polymorphism. As these very weak generic objects seem to play a fundamental role, we discuss them in more detail in Section 4.2 as part of our proposal for a new alignment of terminology.

4 A proposal for a new alignment of terminology

Based on the data and experience of the applications of fibered categories in the theory of triposes, polymorphism, and universes in type theory and algebraic set theory, we may now proceed with some confidence to propose a new alignment of terminology for generic objects that better reflects fibered categorical practice. In this section, we distinguish our proposed usage from that of other authors by underlining.

We will define several notions of generic object in conceptual order rather than in order of strength; in Table 1 we summarize our terminology in order of increasing strength, and compare it to several representatives from the literature.

4.1 Generic objects; skeletal and gaunt

Let $p : \mathscr{E} \longrightarrow \mathscr{B}$ be a fibered category. Our most basic definition below corresponds to the **weak generic objects** of Jacobs [Jac99].

Definition 4.1. A generic object for $p : \mathscr{E} \longrightarrow \mathscr{B}$ is defined to be an object $T \in \mathscr{E}$ such that for any $X \in \mathscr{E}$ there exists a cartesian morphism $X \longrightarrow T$.

Definition 4.2. A generic object $T \in \mathscr{E}$ is called skeletal when for any $X \in \mathscr{E}$, there exists a unique $pX \longrightarrow pT$ such that there exists a cartesian map $X \longrightarrow T$ lying over $pX \longrightarrow pT$.

Definition 4.3. A generic object $T \in \mathscr{E}$ is called gaunt when for any $X \in \mathscr{E}$, any two cartesian morphisms $X \longrightarrow T$ are equal.

Scholium 4.4. Our skeletal and gaunt generic objects are the same as Jacobs' **generic objects** and **strong generic objects** respectively. Our terminology is inspired by a comparison between the properties of an internal category $\mathbb{C}$ and its externalization: in particular, the generic object over $\mathbb{C}_0$ is skeletal when $\mathbb{C}$ is a skeletal category and it is gaunt when $\mathbb{C}$ is a gaunt category. Unfortunately, Phoa [Pho92] has used the word "skeletal" to describe what we call gaunt generic objects; but this usage accords with *op. cit.*'s unconventional definition of a *skeletal category*: usually a skeletal category is one in which any two isomorphic objects are equal, but Phoa defines it to be one in which the only isomorphisms are identity maps. Thus Phoa's skeletal categories are what we would refer to as *gaunt* categories.

4.2 Weak generic objects and stack completions

Let $\mathscr{B}$ be a regular category and let $p : \mathscr{E} \longrightarrow \mathscr{B}$ be a *stack* for the regular topology.

Definition 4.5 (Frey [Fre13, Definition 2.3.5])**.** A cartesian morphism $X \longrightarrow Y$ in $\mathscr{E}$ is called *cover-cartesian* when $pX \twoheadrightarrow pY$ is a regular epimorphism.

Definition 4.6. A weak generic object for $p : \mathscr{E} \longrightarrow \mathscr{B}$ is defined to be an object $T \in \mathscr{E}$ such that for all $X \in \mathscr{E}$ there exists a cartesian span $X \leftarrow \tilde{X} \rightarrow T$ where $\tilde{X} \longrightarrow X$ is cover-cartesian.

Scholium 4.7. Our definition and terminology agrees with that Streicher [Str05, (5.2)] and Battenfield [Bat08, Definition 6.3.7], differing only in that both of the cited works specialize the definition for Bénabou-definable full internal subcategories of topoi, *i.e.* full subfibrations that are are stacks. Very similar to our definition is the *representability axiom* (S2) for classes of small maps in algebraic set theory [JM95, Definition 1.1], except that *op. cit.* do not require $p\tilde{X} \twoheadrightarrow pX$ to be regular.

Remark 4.8. The notion of weak generic object defined above should be thought of an *internal* version of the property of being a generic object. Indeed, our explicit definition ought to be the translation of ordinary genericity through a generalization of Shulman's stack semantics [Shu10] that is stated for stacks other than the fundamental fibration $\mathbf{P}_{\mathscr{B}} = \mathscr{B}^{\rightarrow} \longrightarrow \mathscr{B}$. The same (informal) translation is used by Hyland, Robinson, and Rosolini [HRR90] to correctly define *weak equivalences* and *weak completeness* for categories fibered over a regular category.

Weak generic objects in the sense of Definition 4.6 arise very naturally.

Example 4.9. Let $\pi : E \longrightarrow U$ be a morphism in a finitely complete regular category $\mathscr{B}$; then the class of maps arising as pullbacks of $\pi : E \longrightarrow U$ determines a full subfibration

$[\pi] \subseteq \mathbf{P}_{\mathscr{B}}$ of the fundamental fibration $\mathbf{P}_{\mathscr{B}} = \mathscr{B}^{\to} \longrightarrow\!\!\triangleright\, \mathscr{B}$ for which $\pi : E \longrightarrow U$ is a generic object. The *stack completion* of $[\pi]$ is a weakly (but not necessarily strongly) equivalent full subfibration $\{\pi\} \subseteq \mathbf{P}_{\mathscr{B}}$, and it can be computed like so: an object of $\{\pi\}_I$ is a morphism $X \longrightarrow I$ such that there exists a regular epimorphism $\tilde{I} \twoheadrightarrow I$ such that the pullback $X \times_I \tilde{I} \longrightarrow \tilde{I}$ lies in $[\pi]_{\tilde{I}}$. In other words, we have the pullback squares in the following configuration:

$$\begin{array}{ccccc}
X & \twoheadleftarrow & X \times_I \tilde{I} & \longrightarrow & E \\
\downarrow & \llcorner & \downarrow & \lrcorner & \downarrow \\
I & \twoheadleftarrow & \tilde{I} & \longrightarrow & U
\end{array}$$

Unless $[\pi]$ was already a stack, it is not necessarily the case that $E \longrightarrow U$ is a generic object for the stack completion $\{\pi\}$. But Streicher [Str05] points out that $E \longrightarrow U$ is nevertheless a weak generic object for $\{\pi\}$, essentially by definition.

Scenarios of the form described in Example 4.9 are easy to come by; a canonical example is furnished by the modest objects of a realizability topos as described by Hyland, Robinson, and Rosolini [HRR90], with critical implications to the denotational semantics of polymorphism.

Example 4.10 (Hyland, Robinson, and Rosolini [HRR90])**.** Let **Eff** be the *effective topos* [Hyl82], and let N be its object of realizers, *i.e.* the partitioned assembly given by $\mathbb{N}$ such that $n \Vdash m \in \mathsf{N} \Leftrightarrow m = n$. Write $\mathsf{P} \subseteq \Omega^{\mathsf{N}\times\mathsf{N}}$ for the the object of $\neg\neg$-closed partial equivalence relations on N and $\mathsf{P}' \in \mathbf{Eff} \downarrow \mathsf{P}$ for the *generic* $\neg\neg$*-closed subquotient* of N; internally, this is the subquotient $R : \mathsf{P} \vdash \{x : \mathsf{N} \mid x\ R\ x\}/R$.

The family $\pi : \mathsf{P}' \longrightarrow \mathsf{P}$ then induces a full subfibration $[\pi] \subseteq \mathbf{P}_{\mathbf{Eff}}$ spanned by morphisms that arise by pullback from π; an element $[\pi]_I$ is an object at stage I that is the subquotient of N by some partial equivalence defined at stage I. The fibration $[\pi] \longrightarrow\!\!\triangleright\, \mathbf{Eff}$ is small with generic object π, but it is *not* complete. Although for every **Eff**-indexed diagram $\mathbb{C} \longrightarrow [\pi]$ there "exists" in the internal sense a limit, we cannot globally choose this limit. This situation is referred to by Hyland, Robinson, and Rosolini [HRR90] as *weak completeness* as opposed to *strong completeness*.

In contrast, we may consider the stack completion $\{\pi\}$ of $[\pi]$. An element of $\{\pi\}_I$ is given by an object E at stage I such that there "exists" (in the internal sense) a partial equivalence relation that it is the quotient of — externally, this means that there is a regular epimorphism $\tilde{I} \longrightarrow I$ such that $\tilde{I}^*E$ is the subquotient of N by some partial equivalence relation defined at stage $\tilde{I}$. The stack $\{\pi\}$ is weakly but not strongly equivalent to $[\pi]$; on the other hand, $\{\pi\}$ is complete in the strong sense. Finally, we observe that $\pi : \mathsf{P}' \longrightarrow \mathsf{P}$ is a weak generic object for the stack completion $\{\pi\}$.

If we pull back $[\pi]$ to along the inclusion $i : \mathbf{Asm} \hookrightarrow \mathbf{Eff}$ of assemblies / $\neg\neg$-separated objects in **Eff**, then we have a *complete* fibered category $i^*[\pi]$ over **Asm** which turns out to be (strongly) equivalent to the familiar fibration $\mathbf{UFam}(\mathbf{PER}) \longrightarrow\!\!\triangleright\, \mathbf{Asm}$ of uniform families of PERs over assemblies. In fact, $i^*[\pi]$ is strongly equivalent to $i^*\{\pi\}$

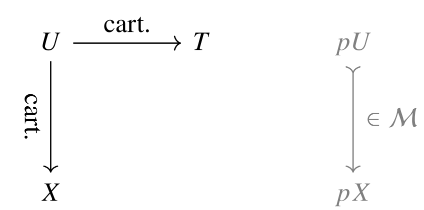
— in other words, over assemblies there is no difference between an object that is locally isomorphic to a subquotient of N and an actual subquotient of N.

There is another way to think of the situation described at the end of Example 4.10, using the observations of Streicher [Str05] on the relationship between Bénabou's notion of *definable class* and stackhood: the class of families of subquotients of N is not definable in **Eff**, but it is definable in **Asm**.

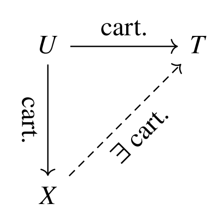
4.3 A new class: *acyclic* generic objects

Inspired by the crucial *realignment* property of type theoretic universes [GSS22], we define a new kind of generic object for an arbitrary fibration that restricts in the case of a full subfibration to a universe satisfying realignment. We refer to Gratzer, Shulman, and Sterling [GSS22] for an explanation of the applications of realignment. In this section, let $p : \mathscr{E} \longrightarrow \mathscr{B}$ be a fibration and let $\mathcal{M}$ be a class of monomorphisms in $\mathscr{B}$.

Definition 4.11. A generic object T for $p : \mathscr{E} \longrightarrow \mathscr{B}$ is called $\mathcal{M}$-*acyclic* when for any span of cartesian maps in $\mathscr{E}$ as depicted below in which $pU \rightarrowtail pX$ lies in $\mathcal{M}$,



there exists a cartesian map $X \longrightarrow T$ making the following diagram commute in $\mathscr{E}$:



Convention 4.12. When $\mathcal{M}$ is understood to be the class of *all* monomorphisms in $\mathscr{B}$, we will simply speak of acyclic generic objects.

All of the examples of acyclic generic objects that we are aware of so far have arisen in the context of the full internal subcategory spanned by a *universe* in the sense of Streicher [Str05], where acyclicity reduces to the *realignment* property studied in detail by Gratzer, Shulman, and Sterling [GSS22].

Scholium 4.13. In the context of universes, the *realignment* or acyclicity condition has referred to as "Axiom (2′)" by Shulman [Shu15], as "strict gluing" by Sterling and Angiuli [SA21], as "stratification" by Stenzel [Ste19], as "strictification" by Orton and Pitts [OP16], and as "alignment" by Awodey [Awo21]. See also Riehl [Rie22] for further discussion.

Origins of the terminology The origin of the term "acyclicity" is explained by Riehl [Rie22] and Shulman [Shu19] in essentially the following way. Let $\mathscr{E} \subseteq \mathbf{P}_{\mathscr{B}}$ be a full subfibration equipped with a generic object T; we will write $\mathbf{Core}(\mathscr{E})$ for the *groupoid core* of $\mathscr{E}$, and for any $I \in \mathscr{B}$ we will write $\mathbf{y}_{\mathscr{B}} I \longrightarrow\!\!\!\triangleright\, \mathscr{B}$ for the discrete fibration whose fiber at $J \in \mathscr{B}$ is the set of morphisms $J \longrightarrow I$. Then we have a canonical morphism of fibered categories $\mathbf{y}_{\mathscr{B}} pT \longrightarrow \mathbf{Core}(\mathscr{E})$ corresponding under the 2-categorical Yoneda lemma to T itself. Realignment for $\mathscr{E}$ is then the property that $\mathbf{y}_{\mathscr{B}} pT \longrightarrow \mathbf{Core}(\mathscr{E})$ has the following right lifting property with respect to any monomorphism $J \rightarrowtail I \in \mathcal{M}$:

$$
\begin{array}{ccc}
\mathbf{y}_{\mathscr{B}} J & \longrightarrow & \mathbf{y}_{\mathscr{B}} pT \\
\downarrow & \overset{\exists}{\nearrow} & \downarrow \\
\mathbf{y}_{\mathscr{B}} I & \longrightarrow & \mathbf{Core}(\mathscr{E})
\end{array}
$$

In a model categorical setting where $\mathcal{M}$ is understood to be the class of cofibrations, the right lifting property above expresses that $\mathbf{y}_{\mathscr{B}} pT \longrightarrow \mathbf{Core}(\mathscr{E})$ is an *acyclic fibration*.

Relationship between acyclic and skeletal generic objects Considering the importance of acyclic generic objects for the semantics of univalent universes, the following observation may explain the real purpose of Voevodsky's startling universe construction involving isomorphism classes of well-ordered simplicial sets [KL21], which produces a skeletal generic object:

Observation 4.14. *A skeletal generic object is always acyclic.*

4.4 Splittings and generic objects

We have shown in Section 2.4 that the correct generalization to non-split fibrations of a **split generic object** in the sense of Jacobs [Jac99] is what we have proposed to call a generic object, *i.e*. the **weak generic object** of *op. cit*. Thus we conclude that the correct relationship can be established between *split* ■*-fibrations* and ■*-fibrations* when re-expressed using our definitions. For example, the following definition expresses the correct relationship between $\lambda 2$-fibrations and split $\lambda 2$-fibrations:

Definition 4.15. A $\lambda 2$-fibration is a fibration with a generic object T, fibered finite products, and simple pT-products and coproducts. A $\lambda 2$-fibration will be called split if all its structure is split.

5 Concluding remarks

For a number of years, the disorder in the variants of generic object has led to a proliferation of subtle differences in terminology between different papers applying fibered categories to categorical logic, type theory, and the denotational semantics of polymorphic types. Based on the kinds of generic object that occur most naturally or have the most utility, we have proposed a unified terminological scheme for generic objects that we believe will meet the needs of scientists working in these areas.

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