

# Bilimits in categories of partial maps

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## Abstract

The closure of chains of embedding-projection pairs (*ep-pairs*) under bilimits in some categories of predomains and domains is standard and well-known. For instance, Scott's  $D_\infty$  construction is well-known to produce directed bilimits of ep-pairs in the category of directed-complete partial orders, and de Jong and Escardó have formalized this result in the constructive domain theory of a topos. The explicit construction of bilimits for categories of predomains and *partial* maps is considerably murkier as far as constructivity is concerned; most expositions employ the constructive taboo that every *lift*-algebra is free, reducing the problem to the construction of bilimits in a category of pointed domains and strict maps. An explicit construction of the bilimit is proposed in the dissertation of Claire Jones, but no proof is given so it remained unclear if the category of dcpos and partial maps was closed under directed bilimits of ep-pairs in a topos. We provide a (Grothendieck)-topos-valid proof that the category of dcpos and partial maps between them is closed under bilimits; then we describe some applications toward models of axiomatic and synthetic domain theory.

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## 1 Preliminaries

(1\*1) In an poset-enriched category  $\mathcal{E}$ , we define an *embedding*  $U \hookrightarrow A$  to be a monomorphism  $\epsilon : U \rightarrow A$  that has a right adjoint, called its *projection*  $\pi : A \rightarrow U$ . Because  $\epsilon \dashv \pi$  and  $\epsilon$  is mono, we have  $\pi \circ \epsilon = \text{id}_U$  and  $\epsilon \circ \pi \leq \text{id}_A$ .

(1\*2) Dually we define a *projection*  $A \rightarrow U$  to be a map  $\epsilon : A \rightarrow U$  that has a monomorphic left adjoint. Embeddings and projections are uniquely determined by each other.

(1\*3) We will write  $\mathcal{E}^E$  for the wide subcategory of  $\mathcal{E}$  spanned by embeddings, and  $\mathcal{E}^P$  for the wide subcategory of  $\mathcal{E}$  spanned by projections. We note that  $\mathcal{E}^P = (\mathcal{E}^E)^{\text{op}}$ .

## 2 Bilimits of directed diagrams in $\text{dcpo}^E$

(2\*1) Let  $\mathcal{I}$  be a directed poset, *i.e.* a filtered category whose hom sets are propositions. Jong and Escardó [JE21] have verified constructively that  $\text{dcpo}^P$  is closed under limits of  $\mathcal{I}^{\text{op}}$ -diagrams, and that, moreover, the cocone obtained from the embeddings of the universal cone is colimiting.

### 2.1 Limits of co-directed diagrams

(2.1\*1) The limit node of a diagram  $D_\bullet : \mathcal{I}^{\text{op}} \rightarrow \text{dcpo}^P$  is explicitly explicitly computed as the following dcpo equipped with the pointwise order:

$$D_\infty := \{ \sigma : \prod_{i \in \mathcal{I}} D_i \mid \forall i \leq j \in \mathcal{I} . \pi_{i \leq j} \sigma_j = \sigma_i \}$$

### 2.1.1 Constructing the limiting cone

(2.1.1\*1) The universal cone  $\{D_\infty\} \rightarrow D_\bullet$  in  $[\mathcal{I}^{\text{op}}, \text{dcpo}^{\text{P}}]$  is defined like so:

$$\begin{aligned}\pi_{i<\infty} : D_\infty &\rightarrow D_i \\ \pi_{i<\infty}\sigma &= \sigma_i\end{aligned}$$

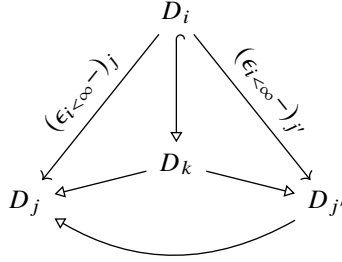
We verify that  $\pi_{i<\infty}$  is a projection by defining its left adjoint explicitly:

$$\begin{aligned}\epsilon_{i<\infty} : D_i &\hookrightarrow D_\infty \\ (\epsilon_{i<\infty}x)_j &= \pi_{j\leq k}\epsilon_{i\leq k}x \quad \text{for } k \geq i, j\end{aligned}$$

Diagrammatically, each component of  $\epsilon_{i<\infty}$  is given like so:

$$D_i \hookrightarrow D_k \longrightarrow D_j$$

We check diagrammatically that  $\epsilon_{i<\infty}$  in fact takes values in  $D_\infty$ :

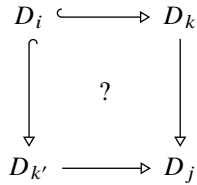


We easily verify that  $\epsilon_{i<\infty}$  is a section of  $\pi_{i<\infty}$ . It remains to check that  $\epsilon_{i<\infty} \circ \pi_{i<\infty} \leq \text{id}_{D_\infty}$ ; because the order is pointwise on  $D_\infty$  it actually suffices to check that  $\pi_{j<\infty} \circ \epsilon_{i<\infty} \circ \pi_{i<\infty} \leq \pi_{j<\infty}$  for each  $j \in \mathcal{I}$ . Fixing  $\sigma \in D_\infty$  we may compute:

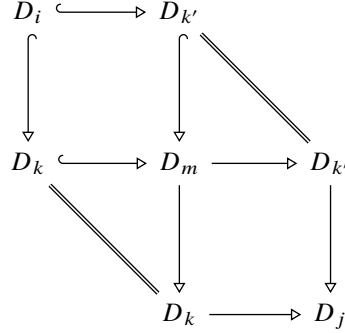
$$\pi_{j<\infty}\epsilon_{i<\infty}\pi_{i<\infty}\sigma = \pi_{j\leq k}\epsilon_{i\leq k}\sigma_i \leq \sigma_j$$

The right-hand inequality holds because each  $\epsilon_{i\leq k} \dashv \pi_{i\leq k}$  is an ep-pair.

(2.1.1\*2) We elaborate on the fact that the embedding  $\epsilon_{i<\infty} : D_i \hookrightarrow D_\infty$  is well-defined: at least one such  $k \geq i, j$  must exist because  $\mathcal{I}$  is directed, but we must also argue that the definition does not depend on the particular choice of  $k$ . Indeed, suppose that we choose two different  $k, k'$  as in the following scenario:



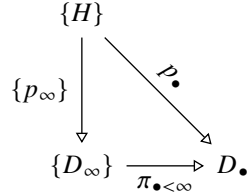
There exists  $m \geq k, k'$  which we use to verify that the diagram commutes:



(2.1.1\*3) Any element  $\sigma \in D_\infty$  is the least upper bound of its family of approximations  $\{\epsilon_{i < \infty} \pi_{i < \infty} \sigma \leq \sigma\}$ . Indeed, fix any  $\sigma' \in D_\infty$  greater than each  $\epsilon_{i < \infty} \pi_{i < \infty} \sigma$ ; we will verify that  $\sigma \leq \sigma'$ . Because the order on  $D_\infty$  is pointwise, it suffices to check that for each  $j \in \mathcal{I}$ , we have  $\pi_{j < \infty} \sigma \leq \pi_{j < \infty} \sigma'$ . Unfolding our assumption, for any  $i, j \in \mathcal{I}$  and  $k \geq i, j$  we have  $\pi_{j \leq k} \epsilon_{i \leq k} \pi_{i < \infty} \sigma \leq \pi_{j < \infty} \sigma'$ . Setting  $i = j = k$ , our goal follows.

## 2.1.2 Universal property of the limiting cone

(2.1.2\*1) Fix another cone  $p_\bullet : \{H\} \rightarrow D_\bullet$  in  $[\mathcal{I}^{\text{op}}, \text{dcpo}^{\text{P}}]$ ; we will exhibit the unique projection  $p_\infty : H \rightarrow D_\infty$  making the following commute:



Note that by duality,  $h$  is equivalently a *cocone* in  $[\mathcal{I}, \text{dcpo}^{\text{E}}]$ .

### 2.1.2.1 Constructing the mediating map

(2.1.2.1\*1) To define  $p_\infty : H \rightarrow D_\infty$  we set each  $(p_\infty)_i : H \rightarrow D_i$  to be simply  $p_i$ ; that this determines an element of  $D_\infty$  is exactly the naturality of  $p_\bullet : \{H\} \rightarrow D_\bullet$ . To see that  $p_\infty$  so-described is a projection, we will explicitly construct its left adjoint  $e_\infty \dashv p_\infty$ . To define the embedding  $e_\infty : D_\infty \hookrightarrow H$ , we will use all of the embeddings  $e_i \dashv p_i$ :

$$\begin{aligned}
 e_\infty &: D_\infty \hookrightarrow H \\
 e_\infty &= \bigvee_{i \in \mathcal{I}} e_i \circ \pi_{i < \infty}
 \end{aligned}$$

To illustrate, we are taking the least upper bound of the following  $\mathcal{I}$ -indexed set of maps:

$$D_\infty \xrightarrow{\pi_{i < \infty}} D_i \xrightarrow{e_i} H$$



### 2.1.2.3 Uniqueness of the mediating map

We must argue that  $p_\infty : H \rightarrow D_\infty$  is the *only* projection map making the following diagram commute:

$$\begin{array}{ccc}
 \{H\} & & \\
 \downarrow \{p_\infty\} & \searrow p_\bullet & \\
 \{D_\infty\} & \xrightarrow{\pi_{\bullet < \infty}} & D_\bullet
 \end{array}$$

This is easily deduced at level of points by considering the universal property of the product  $\prod_{i \in \mathcal{I}} D_i$  of which  $D_\infty$  is a subposet.

## 2.2 As a directed colimit of embeddings

(2.2\*1) As the identification  $\text{dcpo}^E = (\text{dcpo}^P)^{\text{op}}$  proceeds by swapping embeddings for projections, we see that the embeddings corresponding to the limit cone for the diagram  $D_\bullet : \mathcal{I}^{\text{op}} \rightarrow \text{dcpo}^P$  induce a colimiting cone for the equivalent diagram  $D_\bullet : \mathcal{I} \rightarrow \text{dcpo}^E$ .

(2.2\*2) Therefore we have constructed what some refer to as a *bilimit* in the category of embedding-projection pairs over  $\text{dcpo}$ .

## 3 Bilimits of directed diagrams in $\text{pdcpo}^E$

### 3.1 Lifting and partial maps

(3.1\*1) The existence of bilimits of directed diagrams in  $\text{pdcpo}^E$  is folklore; the result easily follows from more general considerations when  $\text{pdcpo} = \text{dcpo}$ , but this identification relies on classical logic. Under slightly different assumptions, a proof is sketched by Fiore [Fio94], and an elementary construction of the bilimit is claimed and not proved by Jones [Jon90] and Jones and Plotkin [JP89]. We will generalize the scenario discussed by *op. cit.* to speak of limits of  $\mathcal{I}^{\text{op}}$ -indexed diagrams of projections in  $\text{pdcpo}^P$ .

(3.1\*2) The *lift* monad  $L : \text{dcpo} \rightarrow \text{dcpo}$  must be defined more carefully than in a classical setting. In particular, we set the carrier set of  $LA$  to be the *partial map classifier* of  $A$ :

$$LA = \sum_{\phi: \Omega} (\phi \Rightarrow A)$$

We will write  $u \downarrow$  to mean  $\pi_1 u = \top$ ; we treat the second projection implicitly in most cases. We impose the order  $u \leq v \Leftrightarrow \forall z : u \downarrow. \exists z' : v \downarrow. u z \leq v z'$ ; in the future we will be less precise and write things like  $u \downarrow \Rightarrow (v \downarrow \wedge u \leq v)$  to mean the same thing. Given a directed family of elements  $\{u_i \in LA \mid i \in \mathcal{I}\}$ , the least upper bound  $\bigvee_{i \in \mathcal{I}} u_i$  is defined to be the least upper bound in  $A$  of the directed family  $\{u_i \in A \mid i \in \mathcal{I} \text{ s.t. } u_i \downarrow\}$ .

**(3.1\*3)** To simplify matters, we note that although partial map  $A \rightarrow B$  is defined to be an ordinary map  $f : A \rightarrow LB$ , we may equivalently define it in terms of the *strict* map  $f^\sharp : LA \multimap LB$ . In other words, the Kleisli category  $\text{pdcpo}$  can be identified with the full subcategory of  $\text{dcpo}$  spanned by *free* L-algebras, *i.e.* objects of the form  $LA$ . In our presentation, we will say that an object of  $\text{pdcpo}$  is a  $\text{dcpo}$  and a morphism from  $A$  to  $B$  is a strict map  $LA \multimap LB$ . The advantage of this presentation of  $\text{pdcpo}$  is that composition of maps is given as in  $\text{dcpo}$  rather than via the Kleisli extension.

## 3.2 Limits of co-directed diagrams of partial projections

**(3.2\*1)** Let  $D_\bullet : I^{\text{op}} \rightarrow \text{dcpo}^{\text{P}}$  be a diagram of partial projections, *i.e.* for each  $i \leq j$  we have a strict map  $\pi_{i \leq j} : LD_j \multimap LD_i$  such that there exists  $\epsilon_{i \leq j} : LD_i \multimap LD_j$  with  $\pi_{i \leq j} \circ \epsilon_{i \leq j} = \text{id}_{LD_i}$  and  $\epsilon_{i \leq j} \circ \pi_{i \leq j} \leq \text{id}_{LD_j}$ . Following Jones and Plotkin [JP89] we define the limit node  $D_\infty$  to be the following  $\text{dcpo}$  equipped with the pointwise order:

$$D_\infty = \{ \sigma : \prod_{i \in I} LD_i \mid (\exists i \in I. \sigma_i \downarrow) \wedge \forall i \leq j \in I. \pi_{i \leq j} \sigma_j = \sigma_i \}$$

### 3.2.1 Constructing the limiting cone

**(3.2.1\*1)** The universal cone  $\{LD_\infty\} \multimap LD_\bullet$  in  $[I^{\text{op}}, \text{pdcpo}^{\text{P}}]$  is defined like so:

$$\begin{aligned} \pi_{i < \infty} &: LD_\infty \multimap LD_i \\ \pi_{i < \infty} &= \mu_{D_i} \circ L\pi_i \end{aligned}$$

where  $\pi_i : \prod_{j \in I} LD_j \rightarrow LD_i$  is the obvious map in  $\text{dcpo}$  and  $\mu_D : L^2D \multimap LD$  is the multiplication map for the lift monad. Alternatively we could have written  $\pi_{i < \infty} = \pi_i^\sharp$  in terms of Kleisli extension.

To see that  $\pi_{i < \infty}$  is a projection, we construct the corresponding embedding.

$$\begin{aligned} \epsilon_{i < \infty} &: LD_i \multimap LD_\infty \\ \epsilon_{i < \infty} x &= L[j \mapsto \pi_{j \leq k} \circ \epsilon_{i \leq k} \circ \eta_{D_i}] \quad \text{for } k \geq i, j \end{aligned}$$

Above we have written  $[j \mapsto \pi_{j \leq k} \circ \epsilon_{i \leq k}]$  for the map  $D_i \rightarrow D_\infty$  determined by the following product cone indexed in  $j \in I$ :

$$D_i \multimap \rightarrow LD_i \multimap \multimap LD_k \multimap \multimap LD_j$$

It is clear that  $\epsilon_{i < \infty}$  in fact takes values in  $LD_\infty$  because the result is at least defined at level  $i$ . We must check that  $\epsilon_{i < \infty}$  is a section of  $\pi_{i < \infty}$ ; considering the uniqueness of L-extensions, it suffices to check that the following commutes:

$$\begin{array}{ccccc} D_i & \xrightarrow{\eta_{D_i}} & LD_i & \xrightarrow{\epsilon_{i < \infty}} & LD_\infty \\ & \searrow \eta_{D_i} & & & \downarrow \pi_{i < \infty} \\ & & & & \downarrow \text{pointwise} \\ & & & & LD_i \end{array}$$

We verify the above using the fact that each  $\epsilon_{i \leq k} \dashv \pi_{i \leq k}$  is an ep-pair:

$$\begin{aligned} \pi_{i < \infty} \circ \epsilon_{i < \infty} \circ \eta_{D_i} &= \pi_{i < \infty} \circ \eta_{D_\infty} \circ (j \mapsto \pi_{j \leq k} \circ \epsilon_{i \leq k} \circ \eta_{D_i}) \\ &= \pi_{i \leq k} \circ \epsilon_{i \leq k} \circ \eta_{D_i} \\ &= \eta_{D_i} \end{aligned}$$

We check that  $\epsilon_{i < \infty} \circ \pi_{i < \infty} \leq \text{id}_{LD_\infty}$  in the same way as for the total case.

**(3.2.1\*2)** Any element  $\sigma \in LD_\infty$  is the least upper bound of its family of approximations  $\{\epsilon_{i < \infty} \pi_{i < \infty} \sigma \leq \sigma\}$ . Fix any other  $\sigma' \in LD_\infty$  greater than each  $\epsilon_{i < \infty} \pi_{i < \infty} \sigma$  to check that  $\sigma \leq \sigma'$ . Assuming  $\sigma = \eta_{D_\infty} \tau$  we must check that there exists some  $\tau' \in D_\infty$  such that  $\sigma' = \eta_{D_\infty} \tau'$  and  $\tau \leq \tau'$ . Because at least one projection of  $\tau$  is defined and  $\sigma'$  is greater than every projection of  $\tau$ , it must be that  $\sigma'$  is defined; therefore, we may fix  $\tau'$  with  $\sigma' = \eta_{D_\infty} \tau'$  and proceed to check that  $\tau \leq \tau'$ , which follows *mutatis mutandis* in the total case **(2.1.1\*3)**.

### 3.2.2 Universal property of the limiting cone

**(3.2.2\*1)** Fix another cone  $p_\bullet : LH \dashv \! \! \dashv LD_\bullet$  in  $[I^{\text{op}}, \text{pdcpo}^{\text{P}}]$ . We will exhibit the unique strict projection  $p_\infty : LH \dashv \! \! \dashv LD_\infty$  making the following commute:

$$\begin{array}{ccc} \{LH\} & & \\ \downarrow \{p_\infty\} & \searrow p_\bullet & \\ \{LD_\infty\} & \xrightarrow{\pi_{\bullet < \infty}} & LD_\bullet \end{array}$$

#### 3.2.2.1 Constructing the mediating map

**(3.2.2.1\*1)** We first define the termination support of  $p_\infty : LH \dashv \! \! \dashv LD_\infty$  as a strict continuous map  $p_\infty \downarrow : LH \dashv \! \! \dashv \Omega$ , which we obtain as the least upper bound of the following directed family of maps indexed in  $i \in I$ :

$$\begin{array}{ccc} LH & \xrightarrow{p_i} \dashv \! \! \dashv & LD_i & \xrightarrow{-\downarrow} \circ & \Omega \\ & \searrow & & \nearrow & \\ & & & & p_i \downarrow \end{array}$$

The least upper bound  $p_\infty \downarrow = \bigvee_{i \in I} p_i \downarrow$  determines a subobject  $\tilde{H} \dashv \! \! \dashv LH$  by pullback



along  $\top$ ; we now construct a *total* map  $\tilde{p}_\infty : \tilde{H} \rightarrow D_\infty$  below:

$$\begin{array}{ccccc}
 D_\infty & \xleftarrow{\tilde{p}_\infty} & \tilde{H} & \longrightarrow & \mathbf{1}_{\text{dcpo}} \\
 \downarrow & & \downarrow \lrcorner & & \downarrow \top \\
 & & (p_\infty \downarrow)^* \top & & \\
 \prod_{i \in \mathcal{I}} \mathsf{L}D_i & \xleftarrow{[i \mapsto p_i]} & \mathsf{L}H & \xrightarrow{p_\infty \downarrow} & \Omega
 \end{array}$$

To see that the factorization above is possible, we note that under  $p_\infty \downarrow$  the product cone  $[i \mapsto p_i]$  really does lie in  $D_\infty$  — because  $p_\infty \downarrow$  is precisely the needed assumption that at least *one* of the  $p_i$  is defined. We now define  $p_\infty$  using the universal property of the partial map classifier:

$$\begin{array}{ccc}
 \tilde{H} & \xrightarrow{\tilde{p}_\infty} & D_\infty \\
 \downarrow \lrcorner & & \downarrow \eta_{D_\infty} \\
 (p_\infty \downarrow)^* \top & & \\
 \downarrow & & \downarrow \\
 \mathsf{L}H & \xrightarrow{\quad p_\infty \quad} & \mathsf{L}D_\infty
 \end{array}$$

By definition,  $p_\infty$  makes the triangle in (3.2.2\*1) commute; it remains to check that  $p_\infty$  is in fact the unique *projection* making that triangle commute.

### 3.2.2.2 The mediating map is a projection

(3.2.2.2\*1) To see that  $p_\infty$  is a projection, we will explicitly construct its left adjoint  $e_\infty : \mathsf{L}D_\infty \hookrightarrow \mathsf{L}H$  as the least upper bound of the following directed set of maps indexed in  $i \in \mathcal{I}$ , just as in (2.1.2.1\*1):

$$\mathsf{L}D_\infty \xrightarrow{\pi_{i < \infty}} \mathsf{L}D_i \xleftarrow{e_i} \mathsf{L}H$$

In other words, we set  $e_\infty = \bigvee_{i \in \mathcal{I}} e_i \circ \pi_{i < \infty}$ .

(3.2.2.2\*2) For  $i \in \mathcal{I}$ , let  $\tilde{H}_i \hookrightarrow \mathsf{L}H$  be the subobject corresponding to the termination support  $p_i \downarrow : \mathsf{L}H \dashrightarrow \mathsf{L}D_i$  such that we have  $\tilde{H} = \bigvee_{i \in \mathcal{I}} \tilde{H}_i$ . Then for each  $i \in \mathcal{I}$ , we have a cartesian square in the following configuration:

$$\begin{array}{ccc}
 D_i & \xrightarrow{\tilde{e}_i} & \tilde{H}_i \\
 \downarrow \eta_{D_i} & \lrcorner & \downarrow \\
 \mathsf{L}D_i & \xrightarrow{e_i} & \mathsf{L}H
 \end{array}$$

That the above exists and is cartesian follows from the pullback lemma, using the fact that each  $e_i$  is a section of  $p_i$ :

$$\begin{array}{ccccc}
 & & \curvearrowright & & \\
 & & \text{---} & & \text{---} \\
 D_i & \xrightarrow{\tilde{e}_i} & \tilde{H}_i & \xrightarrow{\tilde{p}_i} & D_i \\
 \eta_{D_i} \downarrow & \lrcorner & \downarrow & & \downarrow \eta_{D_i} \\
 LD_i & \xleftarrow{e_i} & LH & \xrightarrow{p_i} & LD_i \\
 & & \curvearrowleft & & 
 \end{array}$$

(3.2.2.2\*3) We have a cartesian square in the following configuration:

$$\begin{array}{ccc}
 D_\infty & \xrightarrow{\tilde{e}_\infty} & \tilde{H} \\
 \downarrow & \lrcorner & \downarrow \\
 LD_\infty & \xleftarrow{e_\infty} & LH
 \end{array}$$

To see that this is the case, we see from (3.2.2.2\*2) that the termination support of  $e_\infty$  must be the join of all the subobjects  $\tilde{D}_\infty^i \rightarrow LD_\infty$  defined below:

$$\begin{array}{ccccc}
 \tilde{D}_\infty^i & \xrightarrow{\quad} & D_i & \xrightarrow{\tilde{e}_i} & \tilde{H}_i \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 LD_\infty & \xrightarrow{\pi_{i < \infty}} & LD_i & \xleftarrow{e_i} & LH \\
 & & \eta_{D_i} \downarrow & & 
 \end{array}$$

Observe that  $\tilde{D}_\infty^i$  is the subobject spanned by total elements of  $D_\infty$  whose  $i$ th projection is defined. Therefore the join of all the  $\tilde{D}_\infty^i$  is the subobject of  $LD_\infty$  spanned by total elements of  $D_\infty$  that have at least one defined projection; but we have already ensured that any element of  $D_\infty$  has at least one projection defined. Therefore  $\bigvee_{i \in \mathcal{I}} \tilde{D}_\infty^i$  is  $D_\infty$  itself.

(3.2.2.2\*4) The total map  $\tilde{e}_\infty$  is a section of  $\tilde{p}_\infty$  in the sense that the following triangle commutes:

$$\begin{array}{ccc}
 D_\infty & \xrightarrow{\tilde{e}_\infty} & \tilde{H} \\
 & \searrow & \downarrow \tilde{p}_\infty \\
 & & D_\infty
 \end{array}$$

It suffices to check that each of the following triangles commutes:

$$\begin{array}{ccc}
 D_\infty & \xrightarrow{\tilde{e}_\infty} & \tilde{H} \\
 & \searrow \pi_{i < \infty} & \downarrow \pi_{i < \infty} \circ p_\infty \\
 & & LD_i
 \end{array}$$

This can be seen most easily by chasing an element  $\sigma \in D_\infty$ :

$$\begin{aligned}
 \pi_{i < \infty} \tilde{p}_\infty \tilde{e}_\infty \sigma &= \bigvee \{ \pi_{i < \infty} \tilde{p}_\infty \tilde{e}_j \pi_{j < \infty} \sigma \mid j \in \mathcal{I} \text{ s.t. } \sigma_j \downarrow \} \\
 &= \bigvee \{ \pi_{i \leq k} \pi_{k < \infty} \tilde{p}_\infty \tilde{e}_k \epsilon_{j \leq k} \pi_{j < \infty} \sigma \mid j \in \mathcal{I} \text{ s.t. } \sigma_j \downarrow \} \quad (k \geq i, j) \\
 &= \bigvee \{ \pi_{i \leq k} \tilde{p}_k \tilde{e}_k \epsilon_{j \leq k} \pi_{j < \infty} \sigma \mid j \in \mathcal{I} \text{ s.t. } \sigma_j \downarrow \} \\
 &= \bigvee \{ \pi_{i \leq k} \epsilon_{j \leq k} \pi_{j < \infty} \sigma \mid j \in \mathcal{I} \text{ s.t. } \sigma_j \downarrow \} \\
 &= \bigvee \{ \pi_{i \leq \infty} \epsilon_{j \leq \infty} \pi_{j < \infty} \sigma \mid j \in \mathcal{I} \text{ s.t. } \sigma_j \downarrow \} \\
 &= \pi_{i < \infty} \bigvee \{ \epsilon_{j \leq \infty} \pi_{j < \infty} \sigma \mid j \in \mathcal{I} \text{ s.t. } \sigma_j \downarrow \} \\
 &= \pi_{i < \infty} \sigma
 \end{aligned}$$

**(3.2.2.2\*5)** We must check that  $e_\infty$  is a section of  $p_\infty$ , i.e. the following triangle commutes:

$$\begin{array}{ccc}
 LD_\infty & \xrightarrow{e_\infty} & LH \\
 & \searrow & \downarrow p_\infty \\
 & & LD_\infty
 \end{array}$$

Considering the universal property of the partial map classifier, it suffices to check that both sides of the desired equation can be the bottom map in a cartesian square like the following:

$$\begin{array}{ccc}
 D_\infty & \xlongequal{\quad} & D_\infty \\
 \eta_{D_\infty} \downarrow & \lrcorner & \downarrow \eta_{D_\infty} \\
 LD_\infty & \dashrightarrow & LD_\infty
 \end{array}$$

This obviously holds for the identity map, so it remains to check it for the upper-right



## 4 Other kinds of predomains

It is worth noting that all the arguments above adapt *mutatis mutandis* to other notions of predomains characterized by closure under suprema of more restricted kinds of directed subset; for instance, our arguments establish that both  $\omega\text{cpo}$  and  $\text{p}\omega\text{cpo}$  are closed under bilimits of  $\omega$ -chains of embeddings.

## 5 Axiomatic and synthetic domain theory

### 5.1 Kleisli models of axiomatic domain theory

(5.1\*1) Fiore and Plotkin [FP96] describe a simple recipe to produce models of *axiomatic domain theory* [Fio94] that extend cleanly to sheaf models of *synthetic domain theory* [FR97; Hyl91] supporting recursive types. We recapitulate some definitions from Fiore and Plotkin [FP96] below.

(5.1\*2) Let  $C$  be a category with an initial object and a dominance  $\Sigma$ , along with a  $\Sigma$ -partial map classifier monad  $L$ . An *inductive fixed-point object* in  $C$  is defined by *op. cit.* to be an  $L$ -invariant object  $\sigma : L\bar{\omega} \cong \bar{\omega}$  together with a global element  $\infty : \mathbf{1}_C \rightarrow \bar{\omega}$  such that the following conditions are satisfied

1. The object  $\bar{\omega}$  is the colimit of the following  $\omega$ -chain:

$$\mathbf{0}_C \xrightarrow{!} L\mathbf{0}_C \xrightarrow{L!} L^2\mathbf{0}_C \xrightarrow{L^2!} \dots$$

(We elide the details of the cocone that we are claiming to be universal.)

2. The following diagram commutes:

$$\begin{array}{ccc} \mathbf{1}_C & \xrightarrow{\infty} & \bar{\omega} \\ & \searrow \circlearrowleft & \downarrow \eta_{\bar{\omega}} \\ & & L\bar{\omega} \\ & & \downarrow \cong \\ & & \bar{\omega} \end{array}$$

(5.1\*3) A *monadic base* is defined by *op. cit.* to be a cartesian closed category  $C$  with an initial object, a dominance  $\Sigma$  whose lift monad is written  $L$ , and an inductive fixed point object  $\bar{\omega}$  such that the Eilenberg–Moore category  $C^L$  is closed under tensor products and linear homs.

(5.1\*4) Let  $C$  be a monadic base; then the Kleisli category  $C_L$  is said to be a *Kleisli model of axiomatic domain theory* by *op. cit.* if it is  $C$ -algebraically compact, *i.e.* has *free* algebras for  $C$ -enriched endofunctors.

**(5.1\*5)** Most general results that derive algebraic compactness for  $C_L$  require various assumptions that do not hold in cases of interest; for instance, they may use the fact that  $C_L = C^L$ , and they may require  $C$  to be locally presentable. Neither of these is the case for the category of dcpos in a constructive setting — indeed, the dcpo will never be locally presentable, and the identification of the Kleisli and Eilenberg–Moore categories for the lifting monad is not topos-valid.

**(5.1\*6)** Fiore and Plotkin [FP96] describe a simple recipe to verify the assumptions required by Kleisli models of axiomatic domain theory, quoting a lecture of Plotkin from 1995:

If the Kleisli category  $C_L$  has an enriched zero object and bilimits of  $\omega$ -chains of embedding-projection pairs, then it is  $C$ -algebraically compact.

**(5.1\*7)** Therefore the results contained in this note establish that a Kleisli model of axiomatic domain theory can be fashioned from the dcpos of an arbitrary Grothendieck topos. In particular, for a topos  $\mathcal{E}$  we may consider the (external) category  $\text{dcpo}_{\mathcal{E}}$  of global dcpo-objects in  $\mathcal{E}$  and continuous maps between them. We have a dominance  $\Sigma$  in  $\text{dcpo}_{\mathcal{E}}$  induced by the universal open inclusion  $\mathbf{1} \rightarrow \Omega$ , where  $\Omega$  is the Sierpiński domain corresponding to the subobject classifier of  $\mathcal{E}$ . The corresponding lifting monad for  $\Sigma$  has as a carrier-object the partial map classifier monad of  $\mathcal{E}$ . It is easy to see that the Kleisli category  $\text{pdcpo}_{\mathcal{E}} = (\text{dcpo}_{\mathcal{E}})_L$  has an enriched zero object, and in this note we have proved that it is closed under bilimits  $\omega$ -chains of embedding-projection pairs — indeed, we have proved the stronger result that it is closed under internal bilimits of arbitrary directed posetal diagrams.

## 5.2 The Fiore–Plotkin conservative extension result

**(5.2\*1)** One of the main results of Fiore and Plotkin [FP96] is to establish that every *small* Kleisli model of axiomatic domain theory extends to a sheaf model of an infinitary version of synthetic domain theory (SDT) in which the *well-complete* objects serve as a category of predomains that contains  $C$  in a fully faithful way.

**(5.2\*2)** This sheaf model of synthetic domain theory is obtained in a very simple way; one considers the closed subtopos of  $\tilde{C} \subseteq \widehat{C}$  determined by the subterminal object  $\mathbf{y}(\mathbf{0}_C)$ ; concretely, when  $\mathbf{0}_C$  is strict, this amounts to presheaves taking  $\mathbf{0}_C$  to the terminal object. The lifting monad in  $\tilde{C}$  is obtained by Yoneda extension, and therefore restricts to the original lifting monad.

**(5.2\*3)** The inductive fixed point object  $\bar{\omega} \in C$  becomes the final coalgebra for the lift monad in  $\tilde{C}$ ; the *initial* algebra for the lift monad is denoted  $\omega$ , and comes equipped with a canonical monomorphism  $\omega \rightarrow \bar{\omega}$ . Then a *complete* object is one from whose perspective the lift algebra and the final lift coalgebra appear to coincide, *i.e.* it is internally orthogonal to  $\omega \rightarrow \bar{\omega}$ . A *well-complete* object is an object whose lift is complete. The well-complete objects form a category of predomains, and the Yoneda embedding  $C \hookrightarrow \tilde{C}$  factors through the inclusion of well-complete objects into  $\tilde{C}$ . Among the well-complete objects, the L-algebras support recursion by means of their orthogonality principle; moreover, domain equations can be solved.

(5.2\*4) By restricting  $\text{dcpo}_{\mathcal{E}}$  to small subcategory (e.g. by replacing  $\mathcal{E}$  with the full internal subcategory determined by a topos universe in the sense of Streicher [Str05]), we may therefore compose the results of Fiore and Plotkin [FP96] to obtain models of synthetic domain theory based on “exotic” concrete domain theories; for instance, if  $\mathcal{E}$  is the topos of sheaves on a topological space  $X$ , the domain theory over  $\mathcal{E}$  includes partial maps that terminate only over certain regions of  $X$ .

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