What should a generic object be?

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Abstract

Jacobs has proposed definitions for (weak, strong, split) generic objects for a fibered category; building on his definition of **generic object** and **split generic object**, Jacobs develops a menagerie of important fibrational structures with applications to categorical logic and computer science, including *higher order fibrations*, *polymorphic fibrations*, *$\lambda 2$-fibrations*, *triposes*, and others. We observe that a split generic object need not in particular be a generic object under the given definitions, and that the definitions of polymorphic fibrations, triposes, *etc.* are strict enough to rule out some fundamental examples: for instance, the fibered preorder induced by a partial combinatory algebra in realizability is not a tripos in the sense of Jacobs. We argue for a new alignment of terminology that emphasizes the forms of generic object that appear most commonly in nature, *i.e.* in the study of internal categories, triposes, and the denotational semantics of polymorphic types. In addition, we propose a new class of acyclic generic objects inspired by recent developments in the semantics of homotopy type theory, generalizing the *realignment* property of universes to the setting of an arbitrary fibration.

Contents

1 Introduction

Since the latter half of the 20th century, *fibered category theory* or the *theory of fibrations* has played an important background role in both the applications and foundations of category theory [18, 7]. Fibered categories, also known as fibrations, are a formalism for manipulating categories that are defined *relative* to another category, generalizing the way that ordinary categories can be thought of as being defined relative to the category of sets.

In what sense is ordinary category theory pinned to the category of sets? This can be illustrated by considering the definition of when a category $\mathscr{C}$ "has products":

Definition 1. A category $\mathscr{C}$ has **products** when for any indexed family $\{E_i \in \mathscr{C}\}_{i\in I}$ of objects, there exists an object $\prod_{i\in I} E_i \in \mathscr{C}$ together with a family of morphisms $p_k : \prod_{i\in I} E_i \longrightarrow E_k$ such that for any family of morphisms $h_k : H \longrightarrow E_k$ there exists a unique morphism $h : H \longrightarrow \prod_{i\in I} E_i$ factoring each h_k through p_k.

In the above, the dependency on the category **Set** is clear: the indexing object I is a set. If we had required I to be drawn from a proper subcategory of **Set** (*e.g.* finite sets) or a proper supercategory (*e.g.* classes), the notion of product defined thereby would have been different. The purpose of the formalism of *fibered categories* is to explicitly control the ambient category that parameterizes all indexed notions, such as products, sums, limits, colimits, *etc*.

Remark 2 (Relevance to computer science)**.** The ability to explicitly control the parameterization of products and sums is very important in theoretical computer science, especially for the denotational semantics of *polymorphic types* of the form $\forall\alpha.\tau[\alpha]$. Such a polymorphic type should be understood as the product of all $\tau[\alpha]$ indexed in the "set" of all types, but a famous result of Freyd [13] shows that if a category $\mathscr{C}$ has products of this form *parameterized in* **Set**, then $\mathscr{C}$ must be a preorder. Far from bringing to an early end the study of polymorphic types in computer science, awareness of Freyd's result sparked and guided the search for ambient categories other than **Set** in which to parameterize these products [39, 22, 24]. Fibered category theory provides the optimal language to understand all such indexing scenarios, and the textbook of Jacobs [25], discussed at length in the present paper, provides a detailed introduction to the applications of fibered category theory to theoretical computer science.

1.1 Introduction to fibered categories

Before giving a general definition, we will see the way that fibered categorical language indeed makes parameterization explicit by considering the *prototype* of all fibered categories, the category $\mathbf{Fam}(\mathscr{C})$ of $\mathbf{Set}$-indexed families of objects of a category $\mathscr{C}$.

Construction 3 (The category of families)**.** We define $\mathbf{Fam}(\mathscr{C})$ to be the category of $\mathbf{Set}$-indexed families in $\mathscr{C}$, such that

1. an object of $\mathbf{Fam}(\mathscr{C})$ is a family $\{E_i \in \mathscr{C}\}_{i\in I}$ where I is a set,
2. a morphism $\{F_j\}_{j\in J} \longrightarrow \{E_i\}_{i\in I}$ in $\mathbf{Fam}(\mathscr{C})$ is given by a function $u : J \longrightarrow I$ together with for each $j \in J$ a morphism $\bar{u}_j : F_j \longrightarrow E_{uj}$.

There is an evident functor $p : \mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$ taking $(\{E_i\}_{i\in I})$ to I.

Construction 4 (Fiber categories)**.** For each $I \in \mathbf{Set}$, we may define the *fiber* $\mathbf{Fam}(\mathscr{C})_I$ of $\mathbf{Fam}(\mathscr{C})$ over I to be the category of I-indexed families $\{E_i \in \mathscr{C}\}_{i\in I}$ in $\mathscr{C}$, with morphisms $\{F_i\}_{i\in I} \longrightarrow \{E_i\}_{i\in I}$ given by morphisms $h_i : F_i \longrightarrow E_i$ for each $i \in I$.

More abstractly, the fiber category $\mathbf{Fam}(\mathscr{C})_I$ is the following pullback:

$$\begin{tikzcd}
\mathbf{Fam}(\mathscr{C})_I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Fam}(\mathscr{C}) \arrow[d, "p"] \\
\mathbf{1} \arrow[r, "I"'] & \mathbf{Set}
\end{tikzcd}$$

Construction 5 (Reindexing functors)**.** For any function $u : J \longrightarrow I$, there is a corresponding reindexing functor $u^* : \mathbf{Fam}(\mathscr{C})_I \longrightarrow \mathbf{Fam}(\mathscr{C})_J$ that restricts an I-indexed family into a J-indexed family by precomposition.

With the reindexing functors in hand, can now rephrase the condition that $\mathscr{C}$ has products (Definition 1) in terms of $\mathbf{Fam}(\mathscr{C})$.

Proposition 6. *A category $\mathscr{C}$ has products if and only if for each product projection function $\pi_{I,J} : I \times J \longrightarrow I$, the reindexing functor $\pi_{I,J} : \mathbf{Fam}(\mathscr{C})_I \longrightarrow \mathbf{Fam}(\mathscr{C})_{I\times J}$ has a right adjoint $\prod_{(I,J)} : \mathbf{Fam}(\mathscr{C})_{I\times J} \longrightarrow \mathbf{Fam}(\mathscr{C})_I$ such that the following* Beck–Chevalley *condition holds: for any function $u : K \longrightarrow I$, the canonical natural transformation $u^* \circ \prod_{(I,J)} \longrightarrow \prod_{(K,J)} \circ (u \times \mathsf{id}_J)^*$ is an isomorphism.*

The characterization of products in terms of the category of families may seem more complicated, but it has a remarkable advantage: we can replace $\mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$ with a different functor satisfying similar properties in order to speak more generally of when one category has "products" that are parameterized in another category. The properties that this functor has to satisfy for the notion to make sense are embodied in the definition of a *fibration* or *fibered category;* a functor $\mathscr{E} \longrightarrow \mathscr{B}$ will be called a fibration when it behaves similarly to the functor projecting the parameterizing object from a category of families of objects. We begin with an auxiliary definition of *cartesian morphism*:

Definition 7. Let $p : \mathscr{E} \longrightarrow \mathscr{B}$ and let $E \longrightarrow F$ be a morphism in $\mathscr{E}$, which we depict as follows:

$$\begin{tikzcd}
E \arrow[r] \arrow[d, mapsto] & F \arrow[d, mapsto] \\
pE \arrow[r] & pF
\end{tikzcd}$$

In the diagram above, we say that $E \longrightarrow F$ **lies over** $pE \longrightarrow pF$. We say that $E \longrightarrow F$ is **cartesian** in p when for any morphism $H \longrightarrow F$ in $\mathscr{E}$ and $pH \longrightarrow pE$ such that the former lies over the composite $pH \to pE \to pF$ in $\mathscr{B}$, there exists a unique morphism $H \longrightarrow E$ lying over $pH \longrightarrow pE$ such that $H \longrightarrow F$ lies over the composite $pH \longrightarrow pE$ as depicted:

$$\begin{tikzcd}
H \arrow[r, dashed, "\exists!"] \arrow[d, mapsto] \arrow[rr, bend left] & E \arrow[r] \arrow[d, mapsto] & F \arrow[d, mapsto] \\
pH \arrow[r] & pE \arrow[r] & pF
\end{tikzcd}$$

Remark 8 (Explication of cartesian maps). Returning to our example of the category of families $\mathbf{Fam}(\mathscr{C})$ over **Set**, we can make sense of the notion of a cartesian map. Given a function $u : J \longrightarrow I$ of indexing sets, the reindexing functor u^* takes I-indexed families to J-indexed families. Given an I-indexed family $\{E_i\}_{i\in I}$, we may define a morphism $u^*\{E_i\}_{i\in I} \longrightarrow \{E_i\}_{i\in I}$ in $\mathscr{C}$ whose first component is $u : J \longrightarrow I$ and whose second component is the identity function $E_{uj} \longrightarrow E_{uj}$ at each $j \in J$. The morphism $u^*\{E_i\}_{i\in I} \longrightarrow \{E_i\}_{i\in I}$ is then *cartesian* in $p : \mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$.

Exercise 9. Verify that the morphism $u^*\{E_i\}_{i\in I} \longrightarrow \{E_i\}_{i\in I}$ constructed in Remark 8 is indeed cartesian in $p : \mathbf{Fam}(\mathscr{C}) \longrightarrow \mathbf{Set}$.

There is another way to understand cartesian maps, suggested by the name.

Exercise 10. Let $\mathscr{B}^{\to}$ be the *arrow category* of $\mathscr{B}$, whose objects are morphisms of $\mathscr{B}$ and whose morphisms are commuting squares between them; let $\mathsf{cod} : \mathscr{B}^{\to} \longrightarrow \mathscr{B}$ be the *codomain functor* that projects the codomain of a map $A \longrightarrow B$. Show that a morphism $E \longrightarrow F \in \mathscr{B}^{\to}$ is cartesian if and only if the corresponding square in $\mathscr{B}$ is a pullback square (also called a cartesian square).

The relationship between cartesian morphisms and pullback squares expressed by Exercise 10 suggests a generalization of the conventional "pullback corners" notation to an arbitrary fibration, which we shall use liberally.

Notation 11 (Generalized "pullback corners"). Let $p : \mathscr{E} \longrightarrow \mathscr{B}$ be a functor; when we wish to indicate that a morphism $E \longrightarrow F$ in $\mathscr{E}$ is cartesian over $pE \longrightarrow pF$, we will often draw its display using a "pullback corner" notation as follows:

$$\begin{tikzcd}
E \arrow[r] \arrow[d, mapsto] \arrow[dr, phantom, very near start, "\lrcorner"] & F \arrow[d, mapsto] \\
pE \arrow[r] & pF
\end{tikzcd}$$

Finally we may give the definition of a fibration.

Definition 12. A functor $p : \mathscr{E} \longrightarrow \mathscr{B}$ is called a **fibration** when for any object $E \in \mathscr{E}$ and morphism $B \longrightarrow pE \in \mathscr{B}$ there exists a cartesian morphism $H \longrightarrow E$ lying over $B \longrightarrow pE$. The cartesian morphism is often called the **cartesian lift** of $B \longrightarrow pE$.

Convention 13. We will depict a fibration $p : \mathscr{E} \longrightarrow\!\!\triangleright\, \mathscr{B}$ using triangular arrows. When we wish to leave the functor implicit, we refer to $\mathscr{E}$ as a **fibered category** over $\mathscr{B}$. In the same way that one writes $A \times B$ for the apex of a product diagram, we will often write $\bar{u} : u^*E \longrightarrow E$ for the cartesian lift of $u : B \longrightarrow pE$ as depicted below:

$$\begin{tikzcd}
u^*E \arrow[r, "\bar{u}"] \arrow[d, |->] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, |->] \\
B \arrow[r, "u"'] & pE
\end{tikzcd}$$

In the case of $\mathbf{Fam}(\mathscr{C})$, the existence of cartesian lifts for each $u : B \longrightarrow pE$ corresponds to the *reindexing* functors $u^* : \mathbf{Fam}(\mathscr{C})_{pE} \longrightarrow \mathbf{Fam}(\mathscr{C})_B$.

Exercise 14. Verify that the functor $\mathbf{Fam}(\mathscr{C}) \longrightarrow\!\!\triangleright\, \mathbf{Set}$ is a fibration.

Exercise 15. Conclude from Exercise 10 that the codomain functor $\mathscr{B}^{\rightarrow} \longrightarrow \mathscr{B}$ is a fibration if and only if $\mathscr{B}$ has all pullbacks.

When the codomain functor $\mathscr{B}^{\rightarrow} \longrightarrow \mathscr{B}$ is a fibration, we will refer to it as the *fundamental fibration*, written $\mathbf{P}_{\mathscr{B}}$ following Streicher [55].

1.1.1 Small categories, internal categories

An ordinary category need not have a set of objects — for instance, the category **Set** of all sets has a *proper class* of objects.[1] Likewise, it is possible to find categories such that between two objects there may be a proper class of morphisms (*e.g.* the category of sets and isomorphism classes of spans between them). A category that has hom *sets* is called **locally small**, and a category that has a *set* of objects is called **globally small** by some authors including Jacobs [25]. A category that has both these properties is just called **small**. Small categories are very useful: for instance, if $\mathbb{C}$ is a small category then the category of functors $[\mathbb{C}, \mathbf{Set}]$ is a Grothendieck topos. Functor categories of this kind play an important role in theoretical computer science [*e.g.* 34, 40, 10].

The idea of a (globally, locally) small category can be relativized from **Set** to another category in two *a priori* different ways that ultimately coincide up to equivalence. The simplest and move naïve way to think of a small category $\mathbb{C}$ in a category $\mathscr{B}$ is as an *algebra* for the sorts and operations of the *theory of a category* internal to $\mathscr{B}$, which we develop below; the more sophisticated way is to view $\mathbb{C}$ as a fibration over $\mathscr{B}$ satisfying a generalization of the global and local smallness conditions.

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[1] In this paper, we are somewhat agnostic about set theoretic foundations. Our discussion is compatible with the viewpoint of ZFC, in which classes are taken to be formulas at the meta-level; our discussion is, however, also compatible with other accounts of the "set–class" distinction, such as NBG set theory, MK set theory, or the universe-based approaches of Grothendieck [2] and Mac Lane [30].

Definition 16. Let $\mathscr{B}$ be a category that has pullbacks. An **internal category** or **category object** in $\mathscr{B}$ is given by:

1. an object $\mathbb{C}_0 \in \mathscr{B}$ of *objects*,
2. and an object $\mathbb{C}_1 \in \mathscr{B}$ of *morphisms*,
3. and source and target maps $s, t : \mathbb{C}_1 \longrightarrow \mathbb{C}_0$,
4. and a morphism $i : \mathbb{C}_0 \longrightarrow \mathbb{C}_1$ choosing the identity maps, such that $s \circ i = \mathsf{id}_{\mathbb{C}_0} = t \circ i$,
5. and a morphism $c : \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \longrightarrow \mathbb{C}_1$ choosing composite maps such that $s \circ c = s \circ \pi_1$ and $t \circ c = t \circ \pi_2$,
6. (and several other equations associativity and unit laws of composition)

Observation 17. *When $\mathscr{B}$ =* **Set** *we obtain exactly the ordinary notion of a small category,* i.e. *a small category is the same thing as an* internal category *or* category object *in* **Set**.

In the previous section, we have argued that fibrations are a fruitful way to think about categories defined relative to another category. Indeed, we may view an internal category $\mathbb{C}$ as a fibration via a process called *externalization*. This proceeds in two steps; first we construct a *presheaf of categories* on $\mathscr{B}$, and then we use the *Grothendieck construction* to turn it into a fibration.

Construction 18 (The presheaf of categories associated to an internal category)**.** Let $\mathbb{C}$ be an internal category in $\mathscr{B}$. We may define a presheaf of categories $\mathbb{C}^\bullet : \mathscr{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$ like so:

1. for $I \in \mathscr{B}$, an object of $\mathbb{C}^I$ is given by a morphism $\alpha : I \longrightarrow \mathbb{C}_0$,
2. for $I \in \mathscr{B}$, a morphism $\alpha \longrightarrow \beta \in \mathbb{C}^I$ is given by a morphism $h : I \longrightarrow \mathbb{C}_1$ such that $s \circ h = \alpha$ and $t \circ h = \beta$,
3. for $u : J \longrightarrow I$ in $\mathscr{B}$, the reindexing $u^* : \mathbb{C}^I \longrightarrow \mathbb{C}^J$ is given on both objects and morphisms by precomposition with u.

Construction 19 (The Grothendieck construction)**.** Let $\mathbb{C}^\bullet : \mathscr{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$ be a presheaf of categories; we define its *total category* $\int_{\mathscr{B}} \mathbb{C}^\bullet$ as follows:

1. an object of $\int_{\mathscr{B}} \mathbb{C}^\bullet$ is given by a pair of an object $I \in \mathscr{B}$ and an object $c \in \mathbb{C}^I$,
2. a morphism $(J, c) \longrightarrow (I, d)$ is given by a pair of a morphism $u : J \longrightarrow I \in \mathscr{B}$ and a morphism $c \longrightarrow \mathbb{C}^u d$ in $\mathbb{C}^J$.

There is an evident functor $p : \int_{\mathscr{B}} \mathbb{C}^\bullet \longrightarrow \mathscr{B}$; it is this functor that is referred to as the *Grothendieck construction*.

Exercise 20. Verify that the Grothendieck construction of any presheaf of categories $\mathbb{C}^\bullet : \mathscr{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$ is a fibration.

Observe that the presheaf of categories associated to an internal category is, in each fiber I, the category object in **Set** obtained by restricting along the functor $I : \mathbf{1} \longrightarrow \mathscr{B}$.

Definition 21 (Externalization of an internal category)**.** Let $\mathbb{C}$ be an internal category in $\mathscr{B}$; its **externalization** is defined to be the Grothendieck construction $[\mathbb{C}] := \int_{\mathscr{B}} \mathbb{C}^{\bullet} \rightarrowtriangle \mathscr{B}$ of the associated presheaf of categories (Construction 18).

Remark 22. When $\mathbb{C}$ is an internal category in **Set**, the externalization $[\mathbb{C}]$ is the *family fibration* **Fam**($\mathbb{C}$) described in Construction 3.

As promised we may now isolate the properties of the fibered category $[\mathbb{C}]$ that correspond (up to equivalence) to arising by externalization from an internal category.

Definition 23. A fibered category $\mathscr{E} \rightarrowtriangle \mathscr{B}$ is called **globally small** if there is an object $T \in \mathscr{E}$ such that for any $X \in \mathscr{E}$ there exists a cartesian map $X \longrightarrow T$.

The property of global smallness described above is often phrased as $\mathscr{E} \rightarrowtriangle \mathscr{B}$ having a "generic object" $T \in \mathscr{E}$, but the reason for this paper's existence is that the precise definition of "generic" means in this context is somewhat controversial, and the bulk of the present paper is devoted to justifying the precise meaning for genericity that we have chosen (in agreement with Definition 23).

Definition 24. A fibered category $\mathscr{E} \rightarrowtriangle \mathscr{B}$ is called **locally small** when for any $I \in \mathscr{B}$ and $X, Y \in \mathscr{E}_I$ there exists a span $X \leftarrow H \rightarrow Y$ in the following configuration (with $f : H \longrightarrow X$ cartesian over $pg : H \longrightarrow Y$),

$$\begin{tikzcd}
X \arrow[d, mapsto] & H \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, mapsto] \arrow[dl, phantom, "\lrcorner", very near start] & Y \arrow[d, mapsto] \\
I & pH \arrow[l, "pg"] \arrow[r, "pg"'] & I
\end{tikzcd}$$

such that for any other span $X \leftarrow K \rightarrow Y$ with the same configuration (*i.e.* where $f' : K \longrightarrow X$ is cartesian over $pg' : pK \longrightarrow pY$), there is a unique map $K \longrightarrow H$ making the the following diagram commute:

$$\begin{tikzcd}
 & K \arrow[dl, "f'"'] \arrow[dr, "g'"] \arrow[dd, dashed, "\exists!"] & \\
X & & Y \\
 & H \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Exercise 25 (Intermediate)**.** Let $\mathscr{C}$ be an ordinary category; show that the fibered category **Fam**($\mathscr{C}$) $\rightarrowtriangle$ **Set** is locally small if and only if $\mathscr{C}$ is locally small. Show that $\mathscr{C}$ is equivalent to a small category if and only if **Fam**($\mathscr{C}$) $\rightarrowtriangle$ **Set** is both globally small and locally small.

One of the fundamental results of fibered category theory is that, up to equivalence, global and local smallness in the sense of Definitions 23 and 24 suffice to detect internal categories.

Proposition 26. *A fibration $\mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ is equivalent to the externalization of an internal category $\mathbb{E}$ if and only if it is both globally and locally small.*

Although we do not include the (standard) proof of Proposition 26, it is instructive to understand the object $T \in [\mathbb{E}]$ in the externalization of an internal category $\mathbb{E}$ that renders $[\mathbb{E}]$ globally small. Recalling the definition of the externalization via the Grothendieck construction, we define T to be the pair $(\mathbb{C}_0, \mathsf{id}_{\mathbb{C}_0} : \mathbb{C}_0 \longrightarrow \mathbb{C}_0)$ given by the object of objects and its identity map.

1.1.2 Cleavages and splittings

We briefly recall the definitions of cleavages and splittings for a fibration, as they are relevant to the rest of this paper.

Definition 27. A *cleavage* for a fibration $p : \mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ is a choice $\mathfrak{r}$ of cartesian liftings, assigning to each morphism $u : I \longrightarrow pX \in \mathscr{B}$ in the base an object $\mathfrak{r}_u X \in \mathscr{E}$ over I and a cartesian morphism $\mathfrak{r}_u^\dagger : \mathfrak{r}_u X \longrightarrow pX$ over u.

The data of a cleavage $\mathfrak{r}$ extends for each $u : I \longrightarrow J \in \mathscr{B}$ to a reindexing functor $\mathfrak{r}_u : \mathscr{E}^J \longrightarrow \mathscr{E}^I$.

Definition 28. A cleavage $\mathfrak{r}$ for a fibration $p : \mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ is called *split* when the assignment of reindexing functors $u \mapsto \mathfrak{r}_u$ strictly preserves identities and compositions.

A fibration equipped with a cleavage is called a *cloven fibration*; when this cleavage is split, we speak of *split fibrations*.

Observe that a splitting allows one to view a fibration $\mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ as a presheaf of categories $\mathscr{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$ sending each $I \in \mathscr{B}$ to the fiber $\mathscr{E}^I$. Recalling Construction 18, we see that the externalization of any internal category is split in an obvious way.

1.2 Goals and structure of this paper

Although the property stated in Definition 23, that there exist cartesian morphisms $X \longrightarrow T$ for any X, is the most that can be required for an arbitrary internal category, more restrictive notions of generic object have proved important in practice for different applications. Unfortunately, over the years a number of competing definitions have proliferated throughout the literature — and some of the more established of these definitions lead to false conclusions when taken too literally, as we point out in Section 3.1 in our discussion of Jacobs' mistaken Corollary 9.5.6.

The goal of this paper, therefore, is to argue for a new alignment of terminology for the different forms of generic object that is both internally consistent *and* reflects the use of generic objects in practice. Because generic objects play an important role in several areas of application (categorical logic, algebraic set theory, homotopy type theory, denotational semantics of polymorphism, *etc.*), we believe that we have sufficient evidence today to correctly draw the map.

- In Section 2, we recall the definitions of several variants of generic object by Jacobs [25]; our main observation is that a **split generic object** in the sense of *op. cit.* need not be a **generic object** in the same sense.

- In Section 3, we analyze the consequences of the definitions discussed in Section 2 for the use of generic objects in internal category theory, tripos theory, denotational semantics of polymorphism, algebraic set theory, and homotopy type theory.
- In Section 4, we propose new unified terminology and definitions for all extant forms of generic object (as well as one *new* one). Our proposal is summarized and compared with the literature in Table 1.

2 Four kinds of generic object

We begin by recalling Definition 5.2.8 of Jacobs [25], from which we omit some additional characterizations that will not play a role in our analysis.

> *Consider a fibration* $p : \mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ *and an object* T *in the total category* $\mathscr{E}$. *We call* T *a*
>
> *i)* ***weak generic object*** *if*
>
> $$\forall X \in \mathscr{E}. \exists f : X \longrightarrow T. f \text{ is cartesian}$$
>
> *ii)* ***generic object*** *if*
>
> $$\forall X \in \mathscr{E}. \exists! u : pX \longrightarrow pT. \exists f : X \longrightarrow T. f \text{ is cartesian over } u$$
>
> *iii)* ***strong generic object*** *if*
>
> $$\forall X \in \mathscr{E}. \exists! f : X \longrightarrow T. f \text{ is cartesian}$$

Jacobs [25] then defines **split generic objects** for split fibrations in Definition 5.2.1, paraphrased below:

> A split fibration $p : \mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ has a **split generic object** if there is an object $\Omega \in \mathscr{B}$ together with natural isomorphism $\theta : \mathscr{B}(-, \Omega) \longrightarrow \mathsf{ob}\,\mathscr{E}_\bullet$ in $[\mathscr{B}^{\mathsf{op}}, \mathbf{Set}]$, where the presheaf $\mathsf{ob}\,\mathscr{E}_\bullet$ is defined using the splitting.

A useful characterization of **split generic objects** is given in Lemma 5.2.2 of *op. cit.*:

> *A split fibration* $p : \mathscr{E} \longrightarrow\!\!\triangleright \mathscr{B}$ *has a* ***split generic object*** *if and only if there is an object* $T \in \mathscr{E}$ *with the property that* $\forall X \in \mathscr{E}. \exists! u : pX \longrightarrow pT. u^*T = X$. *[25]*

Comment 29. The **weak** and **strong generic objects** of Jacobs [25] are referred to by Phoa [36] as *generic objects* and *skeletal generic objects*. Phoa [36] does not consider the intermediate notion. On the other hand, Phoa [36] defines *strict generic objects* relative to an arbitrary (non-split) cleavage; a **split generic object** is indeed a *strict generic object* in the sense of Phoa, but even for a split cleavage, a *strict generic object* need not be a **split generic object**. We will discuss Phoa's terminology more in Comment 54.

2.1 Separating generic objects from strong generic objects

Jacobs [25] notes that generic and strong generic objects coincide in fibered preorders, but they may differ otherwise — the difference emanating from the presence of non-trivial vertical automorphisms.

Example 30. Let G be a group containing two distinct elements $u \neq v$, and let $\mathbf{B}G$ be the groupoid with a single object whose hom set is G itself. The family fibration $\mathbf{Fam}(\mathbf{B}G) \rightarrowtriangle \mathbf{Set}$ is a fibered category whose objects are just sets, but such that a morphism $f : I \longrightarrow J \in \mathbf{Fam}(\mathbf{B}G)$ is a pair (f, x) of a function $f : I \longrightarrow J \in \mathbf{Set}$ together with a generalized element $x : I \longrightarrow G$. Moreover, every morphism in $\mathbf{Fam}(\mathbf{B}G)$ is cartesian as $\mathbf{B}G$ is a groupoid. The unique object $T \in \mathbf{Fam}(\mathbf{B}G)_{1_{\mathbf{Set}}}$ is clearly **generic**, but not **strong generic**. Indeed, we have two distinct cartesian morphisms $T \longrightarrow T$ given by the pairs $(\mathsf{id}_{1_{\mathbf{Set}}}, u) \neq (\mathsf{id}_{1_{\mathbf{Set}}}, v)$.

Example 31. Another class of examples comes from considering skeleta of full subcategories of **Set**. For instance, one may take the skeleton of a Grothendieck universe and then externalize to obtain a fibration that has a **generic object** that is not **strong**.

2.2 A split generic object need not be a generic object

Construction 32 (The canonical splitting of the externalization)**.** The externalization of an internal category is split in a canonical way: given $(I, c) \in [\mathbb{C}]$ and $u : J \longrightarrow I$, we choose $u^*(I, c) = (J, c \circ u)$. The cartesian morphism $u^*(I, c) \longrightarrow (I, c)$ is given by the pair $(u, \mathsf{id} \circ c \circ u)$ where $\mathsf{id} : \mathbb{C}_0 \longrightarrow \mathbb{C}_1$ is the generic identity morphism.

Construction 33 (Weak, split generic objects in the externalization)**.** The externalization $p : [\mathbb{C}] \rightarrowtriangle \mathscr{B}$ has a **weak generic object** $T = (\mathbb{C}_0, \mathsf{id}_{\mathbb{C}_0})$. Relative to the splitting of $[\mathbb{C}]$ from Construction 32, the **weak generic object** T is also a **split generic object**.

The **weak generic object** of the externalization of an internal category defined in Construction 33 obviously need not be a **strong generic object**, but it may be more surprising to learn that it also need not be a **generic object** at all. This can happen, for instance, when the internal category $\mathbb{C}$ has two distinct isomorphic objects; the following concrete example illustrates the problem:

Example 34 (A split generic object that is not a generic object)**.** Let U be a set of sets containing two distinct elements A, B with the same cardinality, and $\mathbf{Set}_U \subseteq \mathbf{Set}$ be the full subcategory of **Set** spanned by U. Then the family fibration $\mathbf{Fam}(\mathbf{Set}_U) \rightarrowtriangle \mathbf{Set}$ has a **split generic object** T given by the pair $T = (U, \mathsf{id}_U)$, but T is nonetheless not a **generic object**. Indeed, we have two cartesian morphisms $(1_{\mathbf{Set}}, A) \longrightarrow T$ lying over distinct elements $A \neq B : 1_{\mathbf{Set}} \longrightarrow U$ respectively.

Corollary 35. *A split generic object is not necessarily a generic object.*

2.3 Generic objects from weak generic objects

We have seen that the externalization of an internal category $\mathbb{C}_0$ in $\mathscr{B}$ has an obvious **weak generic object** that is also split; the **weak generic object** $T \in [\mathbb{C}]$

is simply the identity map $\mathsf{id}_{\mathbb{C}_0} : \mathbb{C}_0 \longrightarrow \mathbb{C}_0$ in $\mathscr{B}$. Nonetheless, in some cases the externalization of an internal category may have a **generic object** T' that embodies the *skeleton* of $\mathbb{C}$ as in Example 31, but T' is usually different from the **weak generic object** T described above.

Construction 36 (Computing the skeleton of a small category)**.** Suppose that $\mathscr{B} = \mathbf{Set}$ and thus $\mathbb{C}$ is an ordinary small category. Then we may consider the quotient $\mathbb{C}_0/\cong$ of the objects of $\mathbb{C}$ under isomorphism; in other words, this is the set of isomorphism classes of $\mathbb{C}$-objects. Using the axiom of choice, we may arbitrarily choose a section $s : \mathbb{C}_0/\cong \longrightarrow \mathbb{C}_0$ to the quotient map; moreover, we may choose a function associating to each $u \in \mathbb{C}_0$ an isomorphism $u \cong s[u]_{/\cong}$.

Lemma 37. *The pair $T' = (\mathbb{C}_0/\cong, s)$ is a **generic object** for* **Fam**($\mathbb{C}$)*.*

Proof. Fixing $(I, c) \in \mathbf{Fam}(\mathbb{C})$ we must choose a unique $u : I \longrightarrow pT'$ such that there exists a cartesian map $(I, c) \longrightarrow T$ lying over u. We choose $u(i) = [c(i)]_{/\cong}$, taking each index $i \in I$ to the isomorphism class of $c(i)$.

1. First of all, it is clear that there exists a cartesian map lying over u in the correct configuration.
2. Fixing $v : I \longrightarrow pT'$ such that there exists a cartesian map $(I, c) \longrightarrow T$ lying over v, it remains to show that $v = u$. This follows because such a cartesian map ensures that v and u are the same family of isomorphism classes of objects. □

Lemma 38. *If the T' defined above is a **split generic object**, then $\mathbb{C}$ is skeletal.*

Proof. We have already seen that T is a **split generic object**; but **split generic objects** are unique up to unique isomorphism, hence if T' is split then we must have an isomorphism $T \cong T'$. □

By analogy with Example 31, we next relate **strong genericity** of T to a property of $\mathbb{C}$, which we shall refer to as *essential gauntness*.

Definition 39. A category is called *gaunt* when any isomorphism in that category is the identity.

Definition 40. A category is called *essentially gaunt* when it is equivalent to a gaunt category.

Simon Henry has made the following observation:[2]

Observation 41. *A category is essentially gaunt if and only if any* automorphism *in that category is the identity.*

Lemma 42. *We may also observe that category is gaunt if and only if it is skeletal and essentially gaunt.*

Proof. If $\mathbb{C}$ be gaunt, it is obviously both essentially gaunt and skeletal. Conversely, if $\mathbb{C}$ is essentially gaunt and skeletal, given any isomorphism $f : D \cong C$ we have $D = C$ and this f is an automorphism, which (by gauntness) is the identity. □

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[2] Comment on the MathOverflow thread entitled *Name for 'Category without nontrivial automorphisms'?*, `https://mathoverflow.net/q/370764`.

Comment 43. The term *gaunt* is used by Barwick and Schommer-Pries [4], nLab [33], Univalent Foundations Program [56]; Johnstone [27] has referred in passing to the same notion as *stiffness*.

Lemma 44. *If the T' defined above is a* ***strong generic object****, then $\mathbb{C}$ is essentially gaunt in the sense of Definition 40.*

Proof. Let $f : c \longrightarrow c$ be an automorphism in $\mathbb{C}$, *i.e.* a vertical isomorphism in $\mathbf{Fam}(\mathbb{C})_{1_{\mathbf{Set}}}$. Since T' is **strong generic**, there exists a unique cartesian morphism $(1_{\mathbf{Set}}, c) \longrightarrow T'$; this means that there is a unique element $[c] \in \mathbb{C}_0/\cong$ and a unique isomorphism $h : c \longrightarrow s[c]$. Writing $\phi_c : c \cong s[c]$ for the (globally) chosen isomorphism, we have $f; \phi_c = h = \phi_c$ and hence $f = \mathsf{id}_c$, so $\mathbb{C}$ is essentially gaunt. □

Thus we conclude that although the family fibration $\mathbf{Fam}(\mathbb{C})$ over $\mathbf{Set}$ of a small category $\mathbb{C}$ does have a generic object, this generic object cannot be either a **split generic object** or a **strong generic object** except in somewhat contrived scenarios.

2.4 Weak generic objects are the correct generalization of split generic objects

It is clear that any split generic object is in particular a weak generic object; but the converse *also* holds in a certain sense that we make precise below.

Construction 45 (Presheaf of categories)**.** Let $p : \mathscr{E} \longrightarrow\!\!\!\triangleright\, \mathscr{B}$ be a fibered category and let $T \in \mathscr{E}$ be a **weak generic object** for T. We may construct a presheaf of categories $\mathscr{E}^{\bullet} : \mathscr{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$ like so:

1. an object of $\mathscr{E}^I$ is a morphism $\alpha : I \longrightarrow pT$,
2. a morphism $\alpha \longrightarrow \beta$ in $\mathscr{E}^I$ is a given by
    (a) a cartesian map $\bar{\alpha} : A \longrightarrow T$ over α,
    (b) a cartesian map $\bar{\beta} : B \longrightarrow T$ over β,
    (c) a vertical map $h : A \longrightarrow B$ over I,

   such that $(\bar{\alpha}, \bar{\beta}, h)$ is identified with $(\bar{\alpha}', \bar{\beta}', h')$ when h and h' are equal modulo the unique vertical isomorphisms between the cartesian lifts A, B.

That Construction 45 gives rise to a presheaf of categories relies on the axiom of choice. If $\mathscr{E}$ is equipped with a *cleavage* $\mathfrak{r}$, then an alternative construction can be given that does not rely on the axiom of choice:

Construction 46 (Presheaf of categories, cloven)**.** Let $p : \mathscr{E} \longrightarrow\!\!\!\triangleright\, \mathscr{B}$ be a fibered category equipped with a cleavage $\mathfrak{r}$, and let $T \in \mathscr{E}$ be a **weak generic object** for T. We may construct a presheaf of categories $\mathscr{E}^{\bullet} : \mathscr{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$ like so:

1. an object of $\mathscr{E}^I$ is a morphism $\alpha : I \longrightarrow pT$,
2. a morphism $\alpha \longrightarrow \beta$ in $\mathscr{E}^I$ is a vertical map $h : \mathfrak{r}_{\alpha} T \longrightarrow \mathfrak{r}_{\beta} T$ over I.

Construction 47. Let $\mathbb{E} : \mathscr{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$ be a presheaf of categories; then the Grothendieck construction $q : \int \mathbb{E} \longrightarrow\!\!\!\triangleright\, \mathscr{B}$ has a splitting. Given $(J, \alpha) \in \int \mathscr{E}$ and $u : I \longrightarrow J$, the object $u^*(J, \alpha)$ is chosen to be $(I, \alpha \circ u)$ and the cartesian morphism $u^*(J, \alpha) \longrightarrow (J, \alpha)$ over u in $\int \mathscr{E}^\bullet$ is defined to be the pair $(u, \mathsf{id}_{u^*\alpha})$.

Lemma 48. *There is a fibered equivalence from* $\int \mathscr{E}^\bullet$ *to* $\mathscr{E}$ *over* $\mathscr{B}$*.*

Lemma 49. *The pair* $T' = (T, \mathsf{id}_T)$ *is a* ***split generic object*** *in* $\int \mathscr{E}^\bullet \longrightarrow\!\!\!\triangleright\, \mathscr{B}$*.*

Proof. Fixing $X \in \int \mathscr{E}^\bullet$, we must check that there exists a unique $u : pX \longrightarrow pT'$ such that $u^*T' = X$. Unfolding definitions, we fix $I \in \mathscr{B}$ and $\alpha : I \longrightarrow T$ to check that there is a unique $u : I \longrightarrow T$ such that $(I, u) = (I, \alpha)$. Of course, this is true with $u = \alpha$. □

Thus a **weak generic object** for a fibration $\mathscr{E} \longrightarrow\!\!\!\triangleright\, \mathscr{B}$ generates in a canonical way a new *equivalent* **split** fibration $\mathscr{E}' \longrightarrow\!\!\!\triangleright\, \mathscr{B}$ that has a **split generic object**. This is the correspondence between **weak generic objects** and **split generic objects**.

3 Consequences for internal categories, tripos theory, polymorphism, *etc.*

We recall several definitions from Jacobs [25] below:

1. *A* ***higher order fibration*** *is a first order fibration with a* ***generic object*** *and a cartesian closed base category. Such a higher order fibration will be called* ***split*** *if the fibration is split and all of its fibered structure (including the* ***generic object****) is split. [25, Definition 5.3.1]*
2. *A* ***tripos*** *is a higher order fibration* $\mathscr{E} \longrightarrow\!\!\!\triangleright\, \mathbf{Set}$ *over* **Set** *for which the induced products* $\prod_u$ *and coproducts* $\coprod_u$ *along an arbitrary function* u *satisfy the Beck-Chevalley condition. [25, Definition 5.3.3]*
3. *A* ***polymorphic fibration*** *is a fibration with a generic object, with fibered finite products and with finite products in its base category. It will be called* ***split*** *whenever all this structure is split. [25, Definition 8.4.1].*
4. *A* ***polymorphic fibration*** *with* Ω *in the base as a generic object will be called*
   - *(a) a* $\lambda{\to}$*-fibration if it has fibered exponents;*
   - *(b) a* $\lambda 2$*-fibration if it has fibered exponents and also simple* Ω*-products and coproducts;*
   - *(c) a* $\lambda\omega$*-fibration if it has fibered exponents, simple products and coproducts, and exponents in its base category. [25, Definition 8.4.3]*
5. *Let* $\diamond$ *be* $\to, 2, \omega$*. A* $\lambda\diamond$*-fibration will be called* ***split*** *if all of its structure is split. In particular, its underlying polymorphic fibration is split. [25, Definition 8.4.4]*

A **split generic object** need not be a **generic object** as we have seen in Section 2.2, and indeed, is only quite rarely a generic object. Consequently, a **split polymorphic fibration** is not the same thing as a split fibration with a **split generic object** and split fibered finite products. This disorder suggests that a change of definitions is in order, which we propose in Section 4.

3.1 Consequences for internal category theory

The somewhat chaotic status of **generic objects** *vis-à-vis* **split generic objects** has led to an erroneous claim by Jacobs [25] that the externalization of an internal category has a **generic object**. What is actually proved by *op. cit.* in Lemma 7.3.2 is that the externalization of an internal category has a **split generic object**, but this is later claimed (erroneously) to give rise to a **generic object** in the proof of Corollary 9.5.6. Thus the claimed result of Corollary 9.5.6, that a fibration is small if and only if it is globally small (has a **generic object**) and locally small, is mistaken. We suppose that if this problem had been noticed, the definition of **globally small category** by *op. cit.* would have assumed instead a **weak generic object**.

3.2 Consequences for tripos theory

Jacobs [25] sketches in Example 5.3.4 the standard construction of a tripos from a partial combinatory algebra, referring to Hyland et al. [23] for several parts of the construction (including the **generic object**); the standard definition of a tripos involves a **weak generic object** only [23, 38, 37], a discrepancy that has already been noted by Birkedal [8, p. 110].

This difference is quite destructive, as the **weak generic object** of a realizability tripos will not generally be **generic**, as pointed out by Streicher [54]. Thus we conclude that definition of tripos proposed by Jacobs [25] rules out the main examples of naturally occurring triposes.

Remark 50. A final subtlety: over the category of assemblies, the fibration of *regular* subobjects does indeed have a **strong generic object**, but this is not part of the structure of the tripos.

3.3 Consequences for polymorphism

The reception in the community studying polymorphism has been to either avoid or tacitly correct the definition of generic object. For instance, Hermida [19] speaks of **weak generic objects** and **strong generic objects**, and gives the correct definition of $\lambda\!\to$, $\lambda 2$, $\lambda\omega$ fibration in terms of **weak generic objects**. On the other hand Birkedal et al. [9], Johann and Sojakova [26], Sojakova and Johann [46], Ghani et al. [14] deal mainly with the **split generic objects**, and thus do not seem to run into problems. It can be seen, however, that the examples of (split) $\lambda\diamond$-fibrations in the cited works are *not* in fact (split) $\lambda\diamond$-fibrations in the sense of Jacobs [25], because they do not have **generic objects**. Of course, the problem lies with the definitions rather than the examples.

3.4 Consequences for type theory and algebraic set theory

The idea of a *universe* in a category has been abstracted from Grothendieck's universes [2] by way of the contributions of a number of authors including Bénabou [6], Martin-Löf [31], Joyal and Moerdijk [28], Hofmann and Streicher [20], Streicher [53]. In fibered categorical language, a universe usually is a *full subfibration* of some ambient fibered category that has a **weak generic object**.

In certain cases, the **weak generic object** of a universe is also a **generic object** that is nonetheless not **strong**; one example is Voevodsky's construction of a universe of well-ordered simplicial sets in the context of the simplicial model of homotopy type theory, reported by Kapulkin and Lumsdaine [29]: it is **strong generic** for the fibered category of well-ordered families of simplicial sets and order-preserving morphisms, but only **generic** when considering morphisms that need not preserve the well-orderings. On the other hand, universes of propositions, such as the subobject classifier of a topos or the regular subobject classifier of a quasitopos, are the main source of **strong generic objects** in nature.

The theory of universes as applied to *type theory* on the one hand and *algebraic set theory* on the other hand motivates two additional variants of generic object:

1. In applications to type theory, it has been increasingly important for universes to have a **weak generic object** T that satisfies an additional *realignment* property relative to a class of monomorphisms $\mathcal{M}$; in particular, given a span of cartesian maps $U \leftarrowtail X \to T$ where $p(U \leftarrowtail X) \in \mathcal{M}$, we need an extension $U \to T$ factoring $X \to T$ through $U \rightarrowtail X$. This realignment property has proved essential for the semantics of univalent universes in homotopy type theory [29, 42, 43, 17] as well as Sterling's *synthetic Tait computability* [49], an abstraction of Artin gluing that has been used to prove several metatheoretic results in type theory and programming languages [51, 50, 15, 16, 32, 52]. In Section 4.3 we will discuss the generalization of the realignment property to an arbitrary fibered category as the **acyclic generic object**, which lies strictly between **weak generic** and **generic objects**.

2. In algebraic set theory, universes are considered that enjoy a form of generic object that is even weaker than **weak generic object**. For each $E \in \mathscr{E}$ one only "locally" has a morphism into T; the "very weak" generic objects of algebraic set theory seem to relate to **weak generic objects** in the same way that weak completeness relates to strong completeness in the context of stack completions [22, 24], with important applications to the theory of polymorphism. As these very weak generic objects seem to play a fundamental role, we discuss them in more detail in Section 4.2 as part of our proposal for a new alignment of terminology.

4 A proposal for a new alignment of terminology

Based on the data and experience of the applications of fibered categories in the theory of triposes, polymorphism, and universes in type theory and algebraic set theory, we may now proceed with some confidence to propose a new alignment of terminology for generic objects that better reflects fibered categorical practice. In this section, we distinguish our proposed usage from that of other authors by underlining.

We will define several notions of generic object in conceptual order rather than in order of strength; in Table 1 we summarize our terminology, and compare it to several representatives from the literature. In Fig. 1 we compare the strength of the different kinds of generic object.

Our Proposal	Jacobs [25]	Phoa [36]	Hermida [19]	Streicher [53]
weak generic	—	—	—	weak generic
generic	**weak generic**	generic	generic	generic
acyclic generic	—	—	—	—
skeletal generic	**generic**	—	—	—
gaunt generic	**strong generic**	skeletal generic	strong generic	classifying

Table 1: A Rosetta stone for the different terminologies for generic objects.

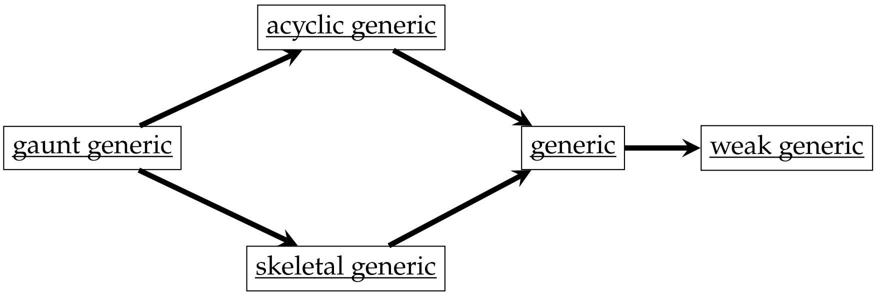

Figure 1: An analysis of the comparability of different notions of generic object, where the rightward direction represents (strictly) decreasing strength.

4.1 Generic objects; skeletal and gaunt

Let $p : \mathscr{E} \longrightarrow \mathscr{B}$ be a fibered category. Our most basic definition below corresponds to the **weak generic objects** of Jacobs [25].

Definition 51. A generic object for $p : \mathscr{E} \longrightarrow \mathscr{B}$ is defined to be an object $T \in \mathscr{E}$ such that for any $X \in \mathscr{E}$ there exists a cartesian morphism $X \longrightarrow T$.

Definition 52. A generic object $T \in \mathscr{E}$ is called skeletal when for any $X \in \mathscr{E}$, there exists a unique $pX \longrightarrow pT$ such that there exists a cartesian map $X \longrightarrow T$ lying over $pX \longrightarrow pT$.

Definition 53. A generic object $T \in \mathscr{E}$ is called gaunt when for any $X \in \mathscr{E}$, any two cartesian morphisms $X \longrightarrow T$ are equal.

Comment 54. Our skeletal and gaunt generic objects are the same as Jacobs' **generic objects** and **strong generic objects** respectively. Our terminology is inspired by a comparison between the properties of an internal category $\mathbb{C}$ and its externalization: in particular, the generic object over $\mathbb{C}_0$ is skeletal when $\mathbb{C}$ is a skeletal category and it is gaunt when $\mathbb{C}$ is a gaunt category. Unfortunately, Phoa [36] has used the word "skeletal" to describe what we call gaunt generic objects; but this usage accords with *op. cit.*'s unconventional definition of a *skeletal category*: usually a skeletal category is one in which any two isomorphic objects are equal, but Phoa defines it to be one in which the only isomorphisms are identity maps. Thus Phoa's skeletal categories would ordinarily be referred to as *gaunt* categories.

To illustrate the comparison between (skeletal, gaunt) categories and (skeletal, gaunt) generic objects respectively, we prove Lemma 55 below.

Lemma 55. *Let $\mathbb{C}$ be an small category and let T be the generic object of* $\mathbf{Fam}(\mathbb{C})$ *over* $\mathbb{C}_0$*.*

1. *$\mathbb{C}$ is skeletal if and only if T is skeletal in the sense of Definition 52;*
2. *$\mathbb{C}$ is gaunt if and only if T is gaunt in the sense of Definition 53.*

Proof. We recall that T is the family $\{C\}_{C\in\mathbb{C}}$.

1. (a) If $\mathbb{C}$ is skeletal, then T is skeletal; fix any family $\{D_i\}_{i\in I}$ and cartesian map $\chi : \{D_i\}_{i\in I} \longrightarrow \{C\}_{C\in\mathbb{C}}$. By assumption, $p\chi$ sends each $i \in I$ to an object $p\chi_i \in \mathbb{C}$ that is isomorphic to D_i; as $\mathbb{C}$ is skeletal, it follows that $p\chi_i = D_i$ and so we conclude that T is skeletal.

   (b) Assume conversely that T is skeletal and fix an isomorphism $f : D_0 \cong D_1$ in $\mathbb{C}$. We have two cartesian maps $h_0, h_1 : \{D_0\} \longrightarrow T$, with one lying over $D_0 : \{*\} \longrightarrow \mathbb{C}_0$ via the identity morphism and the other lying over $D_1 : \{*\} \longrightarrow \mathbb{C}_0$ via f. Since T is skeletal, these two cartesian maps must lie over the same element of $\mathbb{C}_0$, so we have $D_0 = D_1$.

2. (a) If $\mathbb{C}$ is gaunt, then T is gaunt; fix any two cartesian morphisms $h_0, h_1 : \{D_i\}_{i\in I} \longrightarrow \{C\}_{C\in\mathbb{C}}$. Because $\mathbb{C}$ is gaunt and thus skeletal, we know that $ph_0 = ph_1$; thus h_0 assigns to each $i \in I$ an isomorphism $h_{0,i} : D_i \longrightarrow ph_1 i$ which is (by assumption) necessarily the identity. Thus both h_0 and h_1 must send every $i \in I$ to the identity map on D_i and are thus equal.

   (b) Conversely we assume that T is gaunt to check that any isomorphism in $\mathbb{C}$ is an identity map; since T is necessarily also skeletal, it follows by the first case of the present lemma that we may consider just the automorphisms in $\mathbb{C}$, considering Lemma 42. To show that any automorphism $f : D \cong D$ in $\mathbb{C}$ is the identity morphism, we proceed exactly as in the proof of Lemma 44 by observing that the two cartesian maps corresponding to the identity map and the automorphism f respectively are necessarily equal by our assumption that T is gaunt. ☐

4.2 Weak generic objects and stack completions

Let $\mathscr{B}$ be a topos and let $p : \mathscr{E} \longrightarrow \mathscr{B}$ be a fibration. In this section, we are concerned with *stacks* — a 2-dimensional variation on the concept of a sheaf.

Definition 56. Let $\mathscr{X}$ be a site; then a fibration $\mathscr{E} \longrightarrow \mathscr{X}$ is a stack when for any object $X \in \mathscr{X}$ and covering sieve $S \subseteq \mathscr{B} \downarrow X$, the canonical restriction functor $\mathscr{E}_X \longrightarrow \hom_{\mathbf{Fib}_{\mathscr{X}}}(S, \mathscr{E})$ is an equivalence.[3]

Our examples will mainly concern stacks on a topos equipped with its regular topology.

Definition 57 (Frey [12, Definition 2.3.5]). A cartesian morphism $X \longrightarrow Y$ in $\mathscr{E}$ is called *cover-cartesian* when $pX \twoheadrightarrow pY$ is a regular epimorphism.

---

[3] For more on stacks, see The Stacks Project [47] or the tutorial of Vistoli [58].

Definition 58. A weak generic object for $p : \mathscr{E} \rightarrowtriangle \mathscr{B}$ is defined to be an object $T \in \mathscr{E}$ such that for all $X \in \mathscr{E}$ there exists a span of cartesian maps $X \leftarrow \tilde{X} \to T$ where $\tilde{X} \longrightarrow X$ is cover-cartesian.

Comment 59. Our definition and terminology agrees with that of Streicher [53, (5.2)] and Battenfeld [5, Definition 6.3.7], differing only in that both of the cited works specialize the definition for Bénabou-definable full subfibrations of the fundamental fibration of a topos, *i.e.* full subfibrations that are stacks. Our definition is also identical the *representability axiom* (S2) for classes of small maps in algebraic set theory [28, Definition 1.1].

Remark 60. The notion of weak generic object defined above should be thought of as an *internal* version of the property of being a generic object — internal to a fibration that is a stack for the regular topology. Indeed, our explicit definition ought to be the translation of ordinary genericity through a generalization of Shulman's stack semantics [45] that is stated for stacks other than the fundamental fibration $\mathbf{P}_{\mathscr{B}} = \mathscr{B}^{\to} \longrightarrow \mathscr{B}$. The same (informal) translation is used by Hyland et al. [24] to correctly define *weak equivalences* and *weak completeness* for categories fibered over a regular category.

Weak generic objects in the sense of Definition 58 arise very naturally.

Example 61. Let $\pi : E \longrightarrow U$ be a morphism in a topos $\mathscr{B}$; then the class of maps arising as pullbacks of $\pi : E \longrightarrow U$ determines a full subfibration $[\pi] \subseteq \mathbf{P}_{\mathscr{B}}$ of the fundamental fibration $\mathbf{P}_{\mathscr{B}} = \mathscr{B}^{\to} \rightarrowtriangle \mathscr{B}$ for which $\pi : E \longrightarrow U$ is a generic object. The *stack completion* of $[\pi]$ is a weakly (but not necessarily strongly) equivalent full subfibration $\{\pi\} \subseteq \mathbf{P}_{\mathscr{B}}$, and it can be computed like so: an object of $\{\pi\}_I$ is a morphism $X \longrightarrow I$ such that there exists a regular epimorphism $\tilde{I} \twoheadrightarrow I$ such that the pullback $X \times_I \tilde{I} \longrightarrow \tilde{I}$ lies in $[\pi]_{\tilde{I}}$. In other words, we have the pullback squares in the following configuration:

$$\begin{tikzcd}
X \arrow[d] & X \times_I \tilde{I} \arrow[l, two heads] \arrow[r] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d] \\
I & \tilde{I} \arrow[l, two heads] \arrow[r] & U
\end{tikzcd}$$

Unless $[\pi]$ was already a stack, it is not necessarily the case that $E \longrightarrow U$ is a generic object for the stack completion $\{\pi\}$. But Streicher [53] points out that $E \longrightarrow U$ is nevertheless a weak generic object for $\{\pi\}$, essentially by definition.

Scenarios of the form described in Example 61 are easy to come by; a canonical example is furnished by the modest objects of a realizability topos as described by Hyland et al. [24], with critical implications for the denotational semantics of polymorphism.

Example 62 (Hyland et al. [24])**.** Let **Eff** be the *effective topos* [21], and let N be its object of realizers, *i.e.* the partitioned assembly given by $\mathbb{N}$ such that $n \Vdash m \in \mathsf{N} \Leftrightarrow m = n$. Write $\mathsf{P} \subseteq \Omega^{\mathsf{N}\times\mathsf{N}}$ for the the object of $\neg\neg$-closed partial equivalence relations on N and $\mathsf{P}' \in \mathbf{Eff} \downarrow \mathsf{P}$ for the *generic* $\neg\neg$*-closed subquotient* of N; internally, this is the subquotient $R : \mathsf{P} \vdash \{x : \mathsf{N} \mid x \; R \; x\}/R$.

The family $\pi : \mathsf{P}' \longrightarrow \mathsf{P}$ then induces a full subfibration $[\pi] \subseteq \mathbf{P_{Eff}}$ spanned by morphisms that arise by pullback from π; an element $[\pi]_I$ is an object at stage I that is the subquotient of N by some partial equivalence defined at stage I. The fibration $[\pi] \longrightarrow \mathbf{Eff}$ is small with generic object π, but it is *not* complete. Although for every **Eff**-indexed diagram $\mathbb{C} \longrightarrow [\pi]$ there "exists" in the internal sense a limit, we cannot globally choose this limit. This situation is referred to by Hyland et al. [24] as *weak completeness* as opposed to *strong completeness*.

In contrast, we may consider the stack completion $\{\pi\}$ of $[\pi]$. An element of $\{\pi\}_I$ is given by an object E at stage I such that there "exists" (in the internal sense) a partial equivalence relation that it is the quotient of — externally, this means that there is a regular epimorphism $\tilde{I} \longrightarrow I$ such that $\tilde{I}^*E$ is the subquotient of N by some partial equivalence relation defined at stage $\tilde{I}$. The stack $\{\pi\}$ is weakly but not strongly equivalent to $[\pi]$; on the other hand, $\{\pi\}$ is complete in the strong sense. Finally, we observe that $\pi : \mathsf{P}' \longrightarrow \mathsf{P}$ is a weak generic object for the stack completion $\{\pi\}$.

If we pull back $[\pi]$ to along the inclusion $i : \mathbf{Asm} \hookrightarrow \mathbf{Eff}$ of assemblies / $\neg\neg$-separated objects in **Eff**, then we have a *complete* fibered category $i^*[\pi]$ over **Asm** which turns out to be (strongly) equivalent to the familiar fibration $\mathbf{UFam}(\mathbf{PER}) \longrightarrow \mathbf{Asm}$ of uniform families of PERs over assemblies. In fact, $i^*[\pi]$ is strongly equivalent to $i^*\{\pi\}$ — in other words, over assemblies there is no difference between an object that is locally isomorphic to a subquotient of N and an actual subquotient of N.

There is another way to think of the situation described at the end of Example 62, using the observations of Streicher [53] on the relationship between Bénabou's notion of *definable class* and stackhood: the class of families of subquotients of N is not definable in **Eff**, but it is definable in **Asm**.

4.3 A new class: acyclic generic objects

Inspired by the crucial *realignment* property of type theoretic universes [17] that was discussed in Section 3.4(1), we define a new kind of generic object for an arbitrary fibration that restricts in the case of a full subfibration to a universe satisfying realignment. We refer to Gratzer et al. [17] for an explanation of the applications of realignment. In this section, let $p : \mathscr{E} \longrightarrow \mathscr{B}$ be a fibration and let $\mathcal{M}$ be a class of monomorphisms in $\mathscr{B}$.

Definition 63. A generic object T for $p : \mathscr{E} \longrightarrow \mathscr{B}$ is called $\mathcal{M}$-*acyclic* when for any span of cartesian maps in $\mathscr{E}$ as depicted below in which $pU \rightarrowtail pX$ lies in $\mathcal{M}$,

$$\begin{tikzcd}
U \arrow[r, "\text{cart.}"] \arrow[d, "\text{cart.}"'] & T & pU \arrow[d, tail, "\in \mathcal{M}"] \\
X & & pX
\end{tikzcd}$$

there exists a cartesian map $X \longrightarrow T$ making the following diagram commute in

$\mathscr{E}$:

$$\begin{tikzcd}
U \arrow[r, "\text{cart.}"] \arrow[d, "\text{cart.}"'] & T \\
X \arrow[ur, dashed, "\exists\ \text{cart.}"'] &
\end{tikzcd}$$

Convention 64. When $\mathcal{M}$ is understood to be the class of *all* monomorphisms in $\mathscr{B}$, we will simply speak of acyclic generic objects.

All of the examples of acyclic generic objects that we are aware of so far have arisen in the context of the full subfibration spanned by a *universe* in the sense of Streicher [53], where acyclicity reduces to the *realignment* property studied in detail by Gratzer et al. [17].

Remark 65. It is reasonable to ask whether there is a "weak" version of acyclicity that pertains to stack completions in the same way that weak generic objects relate to generic objects, *e.g.* by asking for a cover-cartesian morphism $\tilde{X} \twoheadrightarrow X$ and an extension of $U \times_X \tilde{X} \to U \to T$ along $U \times_X \tilde{X} \longrightarrow \tilde{X}$. It remains somewhat unclear to this author whether this notion is useful, so we do not include it in our proposal.

Comment 66. In the context of universes, the *realignment* or acyclicity condition has been referred to as "Axiom (2')" by Shulman [42], as "strictification" by Orton and Pitts [35], as "stratification" by Stenzel [48], as "alignment" by Awodey [3], and as "strict gluing" by Sterling and Angiuli [50]. See also Riehl [41] for further discussion.

Origins of the terminology The origin of the term "acyclicity" is explained by Shulman [43] and Riehl [41] in essentially the following way. Let $\mathscr{E} \subseteq \mathbf{P}_{\mathscr{B}}$ be a full subfibration equipped with a generic object T; we will write $\mathbf{Core}(\mathscr{E})$ for the *groupoid core* of $\mathscr{E}$, and for any $I \in \mathscr{B}$ we will write $\mathbf{y}_{\mathscr{B}}I \rightarrowtriangle \mathscr{B}$ for the discrete fibration whose fiber at $J \in \mathscr{B}$ is the set of morphisms $J \longrightarrow I$. Then we have a canonical morphism of fibered categories $\mathbf{y}_{\mathscr{B}}pT \longrightarrow \mathbf{Core}(\mathscr{E})$ corresponding under the fibered Yoneda lemma [55, §3] to T itself. Realignment for $\mathscr{E}$ is then the property that $\mathbf{y}_{\mathscr{B}}pT \longrightarrow \mathbf{Core}(\mathscr{E})$ has the following (2-categorical) extension property with respect to any monomorphism $m : J \rightarrowtail I \in \mathcal{M}$, in which we write $\lfloor X \rfloor : \mathbf{y}_{pX} \longrightarrow \mathbf{Core}(\mathscr{E})$ for morphism that we identify with $X \in \mathscr{E}$ under the Yoneda lemma:

$$\begin{tikzcd}
\mathbf{y}_{\mathscr{B}}J \arrow[r, "\mathbf{y}_{\mathscr{B}}\alpha"] \arrow[d, tail, "\mathbf{y}_{\mathscr{B}}m"'] & \mathbf{y}_{\mathscr{B}}pT \arrow[d, "\lfloor T \rfloor"] \\
\mathbf{y}_{\mathscr{B}}I \arrow[r, "\lfloor B \rfloor"'] \arrow[ur, dashed, "\exists \mathbf{y}_{\mathscr{B}}\beta"] & \mathbf{Core}(\mathscr{E})
\end{tikzcd}$$

More formally, for any square as above commuting up to an isomorphism $\phi : \lfloor B \rfloor \circ \mathbf{y}_{\mathscr{B}}m \cong \lfloor T \rfloor \circ \mathbf{y}_{\mathscr{B}}\alpha$, there exists a (not necessarily unique) morphism $\beta : I \longrightarrow pT$ and isomorphisms $\phi_0 : \mathbf{y}_{\mathscr{B}}\beta \circ \mathbf{y}_{\mathscr{B}m} \cong \mathbf{y}_{\mathscr{B}}\alpha$ and $\phi_1 : \lfloor T \rfloor \circ \mathbf{y}_{\mathscr{B}}\beta \cong \lfloor B \rfloor$

such that ϕ factors as the pasting of ϕ_0 and ϕ_1. Moreover, as the boundary of ϕ_0 is discrete, it must be an identity and thus $\beta : I \longrightarrow pT$ extends $\alpha : J \longrightarrow pT$ along $J \rightarrowtail I$ in the (strict) 1-categorical sense. Under the Yoneda lemma, the isomorphism $\phi : \lfloor B \rfloor \circ \mathbf{y}_{\mathscr{B}} m \cong \lfloor T \rfloor \circ \mathbf{y}_{\mathscr{B}} \alpha$ corresponds to an isomorphism $\lceil \phi \rceil : m^* B \cong \alpha^* T$ and the isomorphism $\phi_1 : \lfloor T \rfloor \circ \mathbf{y}_{\mathscr{B}} \beta \cong \lfloor B \rfloor$ corresponds to an isomorphism $\lceil \phi_1 \rceil : \beta^* T \cong B$ that extends $\lceil \phi \rceil$ along $m : J \rightarrowtail I$.

In a model categorical setting where $\mathcal{M}$ is understood to be the class of cofibrations, the extension property above expresses that $\mathbf{y}_{\mathscr{B}} pT \longrightarrow \mathbf{Core}(\mathscr{E})$ is an *acyclic fibration*, whence the terminology.

4.3.1 Incomparability of acyclic and skeletal generic objects

It is not the case that a skeletal generic object need be acyclic. We may consider the following counterexample, which was kindly suggested by one of the anonymous referees of this paper; in what follows, assume that $\mathscr{B}$ is a category with finite limits and that G is a group object in $\mathscr{B}$. As in Example 30, we may regard G as an internal groupoid $\mathbf{B}G$ in $\mathscr{B}$ with a single object $(\mathbf{B}G)_0 = 1_{\mathscr{B}}$, such the hom object $(\mathbf{B}G)_1$ is exactly G. The externalization $[\mathbf{B}G]$ can be seen to have the same objects as $\mathscr{B}$; a morphism $I \longrightarrow J$ in $[\mathbf{B}G]$ is given by a pair of a morphism $f : I \longrightarrow J$ together with a generalized element $u : I \longrightarrow G$. As in any fibered groupoid, every morphism of $[\mathbf{B}G]$ is cartesian. Observe that the object $T = 1_{\mathscr{B}}$ is a skeletal generic object in $[\mathbf{B}G]$; below, we show that T need not be acyclic.

Lemma 67. *The skeletal generic object $T \in [\mathbf{B}G]$ is acyclic if and only if $G \in \mathscr{B}$ has the right lifting property with respect to any element of $\mathcal{M}$.*

Proof. Suppose that T is an acyclic generic object, and fix a monomorphism $m : J \rightarrowtail I \in \mathcal{M}$ together with a generalized element $g : J \longrightarrow G$ in $\mathscr{B}$. We may exhibit a span of cartesian maps in $[\mathbf{B}G]$ as follows, which has a lift by acyclicity:

$$\begin{tikzcd}
J \arrow[r, "(!_J, g)"] \arrow[d, "(m, \epsilon)"'] & T \\
I \arrow[ur, dashed, "(h, \hat{g})"'] &
\end{tikzcd}$$

Above we have exactly a morphism $\hat{g} : I \longrightarrow G$ extending $g : J \longrightarrow G$ along $m : J \rightarrowtail I$. The other direction is analogous. $\square$

It is clear that an acyclic generic object need not be skeletal; the above discussion confirms that acyclic and skeletal generic objects are in fact incomparable.

4.4 Splittings and generic objects

We have shown in Section 2.4 that the correct generalization to non-split fibrations of a **split generic object** in the sense of Jacobs [25] is what we have proposed to call a generic object, *i.e.* the **weak generic object** of *op. cit.* Thus we conclude that the correct relationship can be established between *split ■-fibrations* and *■-fibrations* when re-expressed using our definitions, where ■ ranges over the different kinds of fibered structures (*e.g.* $\lambda 2$-fibration, polymorphic fibration, *etc.*). For example, the following definition expresses the correct relationship between $\lambda 2$-fibrations and split $\lambda 2$-fibrations:

Definition 68. A $\lambda 2$-fibration is a fibration with a generic object T, fibered finite products, and simple pT-products and coproducts. A $\lambda 2$-fibration will be called split if all its structure is split.

4.5 In univalent mathematics

The theory of fibered (1-)categories carries over *mutatis mutandis* to the univalent foundations of mathematics, as shown by Ahrens and Lumsdaine [1]. The pertinent notions of generic object are, however, a bit different in a univalent setting. In particular, gaunt generic objects play a more fundamental role in univalent mathematics than in non-univalent mathematics, because the uniqueness of the classifying squares is becomes a statement of contractibility rather than strict uniqueness. In non-univalent mathematics, uncontrived instances of gaunt generic objects tend to follow the pattern of subobject classifiers in representing a fibered preorder; in univalent mathematics, we also have *object* classifiers (univalent universes) which are gaunt generic objects for the full subfibrations they induce.

Just as in non-univalent mathematics, skeletal generic objects are not expected to play a significant structural role although they may appear when considering the deloopings of groups, as in our 1-categorical examples. The most interesting case is that of acyclic generic objects: the acyclicity property is the correct way to present in non-univalent mathematics a generic object that is gaunt *in the homotopical sense*, as Shulman has recently argued [44]. Thus in a homotopical/univalent setting, one expects to work directly with gaunt generic objects rather than acyclic ones.

5 Concluding remarks

For a number of years, the disorder in the variants of generic object has led to a proliferation of subtle differences in terminology between different papers applying fibered categories to categorical logic, type theory, and the denotational semantics of polymorphic types. Based on the kinds of generic object that occur most naturally or have the most utility, we have proposed a unified terminological scheme for generic objects that we believe will meet the needs of scientists working in these areas.

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