The directed plump ordering

Daniel Gratzer Michael Shulman Jonathan Sterling

March 15, 2022

Abstract

Based on Taylor's hereditarily directed plump ordinals, we define the *directed plump ordering* on W-types in Martin-Löf type theory. This ordering is similar to the plump ordering but comes equipped with non-empty finite joins in addition to the usual properties of the plump ordering.

(0∗0) *Acknowledgment.* This research was supported by the United States Air Force Office of Scientific Research under award number FA9550-21-1-0009.

(0∗1) The theory of plump ordinals [Tay96] has been adapted to Martin-Löf type theory by Fiore, Pitts, and Steenkamp [FPS21] to produce directed well-founded orders suitable for certain transfinite constructions. Given a pair $(A : \mathsf{U}_1, B : A \to \mathsf{U}_1)$, *op. cit.* defines the *plump ordering*: a pair of relations $\leq, \prec$ on a type W of well-founded trees satisfying the following conditions:

1) $\leq$ is reflexive and transitive
2) $\prec$ is transitive and well-founded.
3) If $u \prec v$ then $u \leq v$.
4) If $u \prec v \leq w$ or $u \leq v \prec w$ then $u \prec w$.
5) $(W, \leq)$ has a least element.
6) For each $a : A$, both $\leq$ and $\prec$ have upper-bounds for all $B(a)$-families.

Following Taylor's theory of hereditarily directed plump ordinals [Tay96], we refine this ordering to obtain well-behaved least upper-bounds:

7) Given $u, v : W$ there exists $u \sqcup v$ such that $u \sqcup v \leq w$ if and only if $u, v \leq w$.
8) If $u, v \prec w$ then $u \sqcup v \prec w$.

(0∗2) We have partially formalized our results in Martin-Löf type theory with the UIP principle in the Agda proof assistant [SG22].[1] In particular, all results except the well-foundedness of the list ordering $\sqsubset$ of Section 2 are formalized in Agda.

[1] `http://www.jonmsterling.com/agda-directed-plump-ordering/`.

1 An ordering on W-types

(1∗1) Fix a U_1-container $A \rhd B$ in the sense of Abbott, Altenkirch, and Ghani [AAG05], *i.e.* a pair of a type $A : \mathsf{U}_1$ together with a family of types $B : A \to \mathsf{U}_1$. The *extension* of $A \rhd B$ is the endofunctor $[\![A \rhd B]\!] : \mathsf{U}_1 \to \mathsf{U}_1$ defined like so:

$$\begin{array}{l} \textbf{record}\ [\![A \rhd B]\!]\ (X : \mathsf{U}_1) : \mathsf{U}_1\ \textbf{where} \\ \quad \textbf{constructor}\ (-,-) \\ \quad \mathsf{lbl} : A \\ \quad \mathsf{sub} : B(\mathsf{lbl}) \to X \end{array}$$

The extension of a container is also known as the *polynomial endofunctor* associated to the corresponding morphism $\sum_{x:A} B(x) \longrightarrow A$.

(1∗2) The *initial algebra* for the extension $[\![A \rhd B]\!]$ of a given container can be computed as a W-type in the sense of Martin-Löf [Mar84] consisting of well-founded trees labeled in $a : A$ with subtrees of arity $B(a)$, written $\mathrm{W}_A B : \mathsf{U}_1$. The structure map for this initial algebra is written $\mathsf{ub} : [\![A \rhd B]\!](\mathrm{W}_A B) \longrightarrow \mathrm{W}_A B$, which can be thought of as producing an upper-bound in the subtree order.

(1∗3) Suppose that the container $A \rhd B$ is closed under binary coproducts of shapes in the sense that we have an operation $\hat{+} : A \times A \to A$ such that $B(a_1 \hat{+} a_2) = B(a_1) + B(a_2)$. Given two trees $u, v : \mathrm{W}_A B$, we will write $u \sqcup v$ for $\mathsf{ub}(u.\mathsf{lbl} \hat{+} v.\mathsf{lbl}, [u.\mathsf{sub} \mid v.\mathsf{sub}])$. For a non-empty finite set of trees $\{u_i \mid i \le n\}$, we will write $\bigsqcup_i u_i$ for the corresponding n-ary instance of $\sqcup$.

(1∗4) We may define the following two binary relations $\le, \prec$ on $\mathrm{W}_A B$ as the smallest ones closed under the following rules:

$$\frac{\exists b_1, \dots b_n : B(v.\mathsf{lbl}).\ u \le \bigsqcup_i v.\mathsf{sub}(b_i)}{u \prec v} \qquad \frac{\forall b : B(u.\mathsf{lbl}).\ u.\mathsf{sub}(b) \prec v}{u \le v}$$

Each of **(1∗5)** through **(1∗8)** has been formally verified in Agda.

(1∗5) The relation $\le$ is reflexive.

(1∗6) For any $u, v, w : \mathrm{W}_A B$ we have the following:

1) *Transitivity.* If $u \le v \le w$ then $u \le w$; likewise if $u \prec v \prec w$ then $u \prec w$.
2) *Left flex.* If $u \le v$ and $v \prec w$ then $u \prec w$.
3) *Right flex.* If $u \prec v$ and $v \le w$ then $u \prec w$.

(1∗7) For any $u, v : \mathrm{W}_A B$, if $u \prec v$ then $u \le v$.

(1∗8) Let $\{u_i \mid i \le n\}$ be a non-empty finite family of trees, and let $v : \mathrm{W}_A B$ be a tree; we have $\bigsqcup_i u_i \le v$ if and only if $u_i \le v$ for all $i \le n$. Morever, we have $\bigsqcup_i u_i \prec v$ if $u_i \prec v$ for all $i \le n$.

2 An intermezzo on list orderings

(2∗1) Given a relation $R : A \times A \longrightarrow \Omega$, define the accessibility predicate as the following inductive type:

$$\begin{array}{l} \textbf{data}\ \mathsf{Acc}(R) : A \to \Omega\ \textbf{where} \\ \quad \mathsf{acc} : (a : A) \to ((b : A) \to R(b, a) \to \mathsf{Acc}(R, b)) \to \mathsf{Acc}(R, a) \end{array}$$

A relation is said to be well-founded when all its elements are accessible. Note that a well-founded relation need not be transitive.

(2∗2) We eventually wish to show that $\prec$ is well-founded but prior to this we must introduce a supplementary well-founded ordering. The well-foundedness of $\prec$ will follow from well-founded induction on this secondary ordering.

Fix a type X and a well-founded relation $< : X \times X \to \Omega$ for the remainder of this section. We define a new relation $\sqsubset$ on $\mathsf{List}(X)$:

$$\frac{m \geq 1 \qquad \exists f : \{1 \ldots n\} \to \{1 \ldots m\}.\, \forall i \leq n.\, x_i < y_{f(i)}}{[x_1, \ldots, x_n] \sqsubset [y_1, \ldots, y_m]}$$

We adapt a proof due to Wilfried Buchholz as described by Nipkow [Nip98] to prove that $\sqsubset$ is well-founded.

(2∗3) The empty list is $\sqsubset$-accessible.

(2∗4) If a list is $\sqsubset$-accessible, so too is any permutation.

(2∗5) Fix $y : X$. Suppose for all accessible $l : \mathsf{List}(X)$ and $x < y$, $\mathsf{cons}(x, l)$ is accessible. Then for all accessible $l : \mathsf{List}(X)$, $\mathsf{cons}(y, l)$ is accessible.

Proof. Fix an accessible l and suppose that $n \sqsubset \mathsf{cons}(y, l)$. By definition, there exists a division of n into n_l and n_y such that $n_l \sqsubset l$ and each element of n_y is dominated by y. Because l is accessible, so too is n_l. Therefore, $n_y + n_l$ is accessible by induction on the size of n_y and repeated use of the assumption. Because n is a permutation of $n_y + n_l$, we conclude that n is accessible. □

(2∗6) If $l : \mathsf{List}(X)$ is $\sqsubset$-accessible and $x : X$, then $\mathsf{cons}(x, l)$ is accessible.

Proof. This follows immediately from the **(2∗5)** and $<$-induction on x. □

(2∗7) If $<$ is well-founded, so too is $\sqsubset$.

Proof. Fix $l : \mathsf{List}(X)$. We argue by induction on l that l is accessible. In the base case apply **(2∗3)** and in the inductive step apply **(2∗6)**. □

3 Well-foundedness of the directed plump ordering

(3∗1) Write $\mathsf{List}^+(X)$ for the type of *non-empty* lists. Given an non-empty list $l = [u_0, \ldots, u_n]$, write $\bigsqcup l$ for $\bigsqcup_{i \leq n} u_i$.

(3∗2) Given $l : \mathsf{List}^+(\mathrm{W}_A B)$, if $u \leq \bigsqcup l$ then u is $\prec$-accessible.

Proof. This follows by well-founded induction on the $\sqsubset$-accessibility of l; the details are formalized in Agda. □

(3∗3) The relation $\prec$ is well-founded.

Proof. We must prove that every $u : \mathrm{W}_A B$ is $\prec$-accessible, but this is a consequence of **(3∗2)** setting l to be the singleton list $[u]$; the details are formalized in Agda. □

(3∗4) Summarizing, given a pair $(A : \mathsf{U}_1, B : A \to \mathsf{U}_1)$ together with an operation an operation $\hat{+} : A \times A \to A$ such that $B(a_1 \mathbin{\hat{+}} a_2) = B(a_1) + B(a_2)$ there exists a type $\mathrm{W}_A B$ together with a pair of relations $\leq, \prec : \mathrm{W}_A B \times \mathrm{W}_A B \to \Omega$ satisfying the following conditions:

1) $\leq$ is transitive and reflexive.
2) $\prec$ is transitive and well-founded.
3) If $u \prec v$, then $u \leq v$.
4) If $u \prec v \leq w$ or $u \leq v \prec w$ then $u \prec w$
5) If there exists $a : A$ such that $B(a) = \mathbf{0}$ then $(\mathrm{W}_A B, \leq)$ has a least element.
6) For any $a : A$, both $\leq$ and $\prec$ have upper-bounds for all $B(a)$-families.
7) Given u, v there exists an element $u \sqcup v$ such that $u \sqcup v \leq w$ if and only if $u, v \leq w$.
8) If $u, v \prec w$ then $u \sqcup v \prec w$.

(3∗5) Given a pair $(A : \mathsf{U}_1, B : A \to \mathsf{U}_1)$, define a new pair (C, D) by setting $C = \mathsf{List}(A)$ and specifying D inductively:

$$D([\,]) = \mathbf{0} \qquad D(\mathsf{cons}(a, c)) = B(a) + D(c)$$

Then **(3∗4)** instantiated with this new family shows that $(\mathrm{W}_C D, \leq, \prec)$ satisfies the requirements outlined by **(0∗1)**.

References

[Mar84] Per Martin-Löf. *Intuitionistic type theory. Notes by Giovanni Sambin*. Vol. 1. Studies in Proof Theory. Bibliopolis, 1984, pp. iv+91. ISBN: 88-7088-105-9 (cit. on p. 2).

[Tay96] Paul Taylor. “Intuitionistic sets and ordinals”. In: *The Journal of Symbolic Logic* 61.3 (1996), pp. 705–744. DOI: 10.2307/2275781 (cit. on p. 1).

[Nip98] Tobias Nipkow. *An Inductive Proof of the Wellfoundedness of the Multiset Order*. Exposition of a proof due to Wilfried Buchholz. 1998. URL: https://www21.in.tum.de/~nipkow/Misc/multiset.ps (cit. on p. 3).

[AAG05] Michael Abbott, Thorsten Altenkirch, and Neil Ghani. “Containers: Constructing strictly positive types”. In: *Theoretical Computer Science* 342.1 (2005). Applied Semantics: Selected Topics, pp. 3–27. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2005.06.002 (cit. on p. 2).

[FPS21] Marcelo P. Fiore, Andrew M. Pitts, and S. C. Steenkamp. *Quotients, inductive types, and quotient inductive types*. 2021. arXiv: 2101.02994 [cs.LO] (cit. on p. 1).

[SG22] Jonathan Sterling and Daniel Gratzer. agda-directed-plump-ordering. https://github.com/jonsterling/agda-directed-plump-ordering. 2022 (cit. on p. 1).