A synthetic realignment theorem

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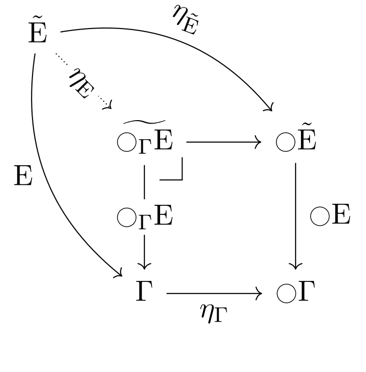
(0·1) Orton and Pitts [OP16] noted that the Hofmann–Streicher universes [HS97] of $\mathrm{Pr}(\mathscr{C})$ support a univalence-like *strictification* principle. Working internally to $\mathrm{Pr}(\mathscr{C})$, let $\phi : \Omega$ be a proposition; for each $\mathrm{A} : \mathscr{U}$ and $\mathrm{A}_\phi : [\phi] \to \mathscr{U}$ together with $f_\phi : \prod_{z:[\phi]}(\mathrm{A} \cong \mathrm{A}_\phi(z))$, there is a type $\mathrm{G} := \mathsf{Glue}_{\mathrm{A}}\,[\phi \to (\mathrm{A}_\phi, f_\phi)] : \mathscr{U}$ together with an isomorphism $g : \mathrm{A} \cong \mathrm{G}$ with the property that under $[\phi]$, we have $\mathrm{G} = \mathrm{A}_\phi$ and $g = f_\phi$ strictly.

(0·2) The first application of the strictification theorem **(0·1)** is to implement a univalent universe in models of cubical type theory [OP16; Ang+19]. More recently, Sterling and Harper [SH20] and Sterling and Gratzer [SG20] have applied the strictification principle of Orton and Pitts to develop a "synthetic Tait computability" over any presheaf topos equipped with a partition into open and closed subtopoi. The purpose of the strictification in this setting is to construct universes of computability families together with codes that lie *directly* over certain syntactic codes, and thereby trivialize the difficult construction of the computability models for dependent type theory.

(0·3) Synthetic Tait computability axiomatizes certain aspects of the situation of *recollement*/gluing of topoi along the "boundary" between a pair of complementary open/closed subtopoi; the application of synthetic Tait computability to prove metatheorems for type theories and programming languages is carried out by finding an open subtopos $\mathcal{X}_{\mathrm{U}} \hookrightarrow \mathcal{X}$ such that the syntactic category of a given type theory can be embedded into $\mathrm{Sh}(\mathcal{X}_{\mathrm{U}})$.

1 Lex idempotent modalities and dependent types

(1·1) Let $\mathcal{X}$ be a topos and let $\mathscr{E} = \mathrm{Sh}(\mathcal{X})$; let $\bigcirc : \mathscr{E} \longrightarrow \mathscr{E}$ be a lex monad on $\mathscr{E}$. In this situation, we may internalize $\bigcirc$ as a modality in the internal type theory of $\mathcal{X}$. Let Γ be a sheaf on $\mathcal{X}$, and let $\mathrm{E} : \mathscr{E}_{/\Gamma}$ be a family of sheaves over Γ; we define $\bigcirc_\Gamma \mathrm{E} : \mathscr{E}_{/\Gamma}$ by functoriality and pullback along the unit as follows:



(1·1·1)

To see that we actually have a type theoretic connective, we must check that the following square commutes up to isomorphism for each $\gamma : \Delta \longrightarrow \Gamma$ in $\mathscr{E}$.

$$\begin{array}{ccc} \mathscr{E}_{/\Gamma} & \xrightarrow{\bigcirc_\Gamma} & \mathscr{E}_{/\Gamma} \\ {\scriptstyle\gamma^*}\big\downarrow & & \big\downarrow{\scriptstyle\gamma^*} \\ \mathscr{E}_{/\Delta} & \xrightarrow[\bigcirc_\Delta]{} & \mathscr{E}_{/\Delta} \end{array} \tag{1·1·2}$$

To check this, we fix $\mathrm{E} : \mathscr{E}_{/\Gamma}$ and compute $\bigcirc_\Delta(\gamma^*\mathrm{E}) : \mathscr{E}_{/\Delta}$; below, the right-hand square is a pullback square because $\bigcirc$ is left exact, and therefore the outer square is also a pullback square.

$$\begin{array}{ccccc} \widetilde{\bigcirc_\Delta\gamma^*\mathrm{E}} & \longrightarrow & \widetilde{\bigcirc\gamma^*\mathrm{E}} & \longrightarrow & \bigcirc\tilde{\mathrm{E}} \\ {\scriptstyle\bigcirc_\Delta\gamma^*\mathrm{E}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\bigcirc\gamma^*\mathrm{E}} & & \big\downarrow{\scriptstyle\bigcirc\mathrm{E}} \\ \Delta & \xrightarrow[\eta_\Delta]{} & \bigcirc\Delta & \xrightarrow[\bigcirc\gamma]{} & \bigcirc\Gamma \end{array} \tag{1·1·3}$$

But the axioms of the monad guarantee that $\bigcirc\gamma \circ \eta_\Delta = \eta_\Gamma \circ \gamma$, so Diagram 1·1·3 is isomorphic to Diagram 1·1·4 below:

$$\begin{array}{ccccc} \widetilde{\gamma^*\bigcirc_\Gamma\mathrm{E}} & \longrightarrow & \widetilde{\bigcirc_\Gamma\mathrm{E}} & \longrightarrow & \bigcirc\tilde{\mathrm{E}} \\ {\scriptstyle\gamma^*\bigcirc_\Gamma\mathrm{E}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\bigcirc_\Gamma\mathrm{E}} & & \big\downarrow{\scriptstyle\bigcirc\mathrm{E}} \\ \Delta & \xrightarrow[\gamma]{} & \Gamma & \xrightarrow[\eta_\Gamma]{} & \bigcirc\Gamma \end{array} \tag{1·1·4}$$

(1·2) Every topos has a hierarchy of type theoretic universes; on a presheaf topos these are obtained as in Hofmann and Streicher [HS97], and over a localization thereof, they are obtained by sheafification as in Streicher [Str05]. Under mild size constraints, this localization process internalizes into the type theory in a certain sense.

When $\dot{\mathscr{U}} \longrightarrow \mathscr{U}$ is the generic family for a given universe, we may ask for the corresponding map $\bigcirc\dot{\mathscr{U}} \longrightarrow \bigcirc\mathscr{U}$ to be $\mathscr{U}$-small in the sense that we have a pullback square in the following configuration:

$$\begin{array}{ccc} \bigcirc\dot{\mathscr{U}} & \dashrightarrow & \dot{\mathscr{U}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \bigcirc\mathscr{U} & \underset{\hat{\bigcirc}}{\dashrightarrow} & \mathscr{U} \end{array}$$

The family $\bigcirc\dot{\mathscr{U}} \longrightarrow \bigcirc\mathscr{U}$ serves as a strict replacement for the universe of $\bigcirc$-modal types: indeed, any small type that is $\bigcirc$-modal has (at least) one code in $\bigcirc\mathscr{U}$. The benefit of this construction over $\{A : \mathscr{U} \mid A \text{ is } \bigcirc\text{-modal}\}$ is that the latter would itself not be modal, whereas the former is. If we make sure that the universe of modal types is modal, we can accomodate hierarchies of universes of modal types. Hereafter, we write $\mathscr{U}_{\bigcirc}$ for $\bigcirc\mathscr{U}$.

The "dependent modality" $\hat{\bigcirc} : \mathscr{U}_{\bigcirc} \longrightarrow \mathscr{U}$ can be seen as the internal version of the *direct image* part of the open immersion determined by $\bigcirc$; the unit $\eta : \mathscr{U} \longrightarrow \mathscr{U}_{\bigcirc}$ acts as the *inverse image* part of the immersion, *i.e.* the sheafification. Composing these, one obtains an internal monad $\mathscr{U} \longrightarrow \mathscr{U}$, the "non-dependent modality".

2 Open and closed modality

(2·1) In the category of sheaves $\mathrm{Sh}(\mathcal{X})$, an open subtopos $\mathcal{X}_{\mathrm{U}}$ is internalized as a proof-irrelevant propositions $\mathrm{U} : \Omega$, called an *open* [AGV72]. Externally, any sheaf E on $\mathcal{X}$ can be thought of as a *étale* topos $\mathcal{X}_{\mathrm{E}} \longrightarrow \mathcal{X}$ lying over $\mathcal{X}$, corresponding under inverse image to the base change $\mathrm{E}^* : \mathrm{Sh}(\mathcal{X}) \longrightarrow \mathrm{Sh}(\mathcal{X})_{/\mathrm{E}}$. Geometrically, the étale topos can be intersected with the open subtopos:

$$\begin{array}{ccc} \mathcal{X}_{\bigcirc\mathrm{E}} & \longrightarrow & \mathcal{X}_{\mathrm{E}} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{X}_{\mathrm{U}} & \hookrightarrow & \mathcal{X} \end{array}$$

The sheaf corresponding to the *étale* map $\mathcal{X}_{\bigcirc\mathrm{E}} \longrightarrow \mathcal{X}$ is the exponential $\bigcirc\mathrm{E} := \mathrm{E}^{\mathrm{U}}$; the assignment $\mathrm{E} \longmapsto \bigcirc\mathrm{E}$ can be seen to be the (lex idempotent) monad induced by the adjunction $j^* \dashv j_*$ where $j : \mathcal{X}_{\mathrm{U}} \hookrightarrow \mathcal{X}$ is the open immersion. As in Section 1, the subtopos becomes an *open modality* [RSS20] on the universes of the type theory.

(2·2) The *closed complement* of $\mathcal{X}_{\mathrm{U}}$ is the topos $\mathcal{X}_{\setminus\mathrm{U}}$ corresponding to the full subcategory of $\mathscr{E}$ spanned by $\bigcirc$-connected sheaves E, in the sense that $\bigcirc\mathrm{E} \cong \mathbf{1}$. The corresponding monad $\bullet : \mathscr{E} \longrightarrow \mathscr{E}$ can be defined as a pushout:

$$\begin{array}{ccc} \mathrm{E} \times \mathrm{U} & \longrightarrow & \mathrm{U} \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{E} & \longrightarrow & \bullet\mathrm{E} \end{array}$$

The monad is immediately seen to be idempotent, and it is seen to be lex as well. One quickly checks that the modality isolates exactly the $\bigcirc$-connected sheaves; following Section 1, we may also internalize $\bullet$ as a modality in the type theory.

(2·3) *The fracture theorem [AGV72; RSS20].* Let $\mathrm{E} : \mathscr{E}$ be any sheaf; the following diagram is a pullback square:

$$\begin{array}{ccc} \mathrm{E} & \xrightarrow{\eta^{\bullet}_{\mathrm{E}}} & \bullet\mathrm{E} \\ {\scriptstyle \eta^{\circ}_{\mathrm{E}}}\downarrow & \lrcorner & \downarrow{\scriptstyle \bullet\eta^{\circ}_{\mathrm{E}}} \\ \bigcirc\mathrm{E} & \xrightarrow[\eta^{\bullet}_{\bigcirc\mathrm{E}}]{} & \bullet\bigcirc\mathrm{E} \end{array}$$

(2·4) The significance of **(2·3)** is that it provides that every sheaf is totally determined by its restriction to the open and closed subtopoi respectively; this is the "microcosmic" view of the fact that the topos $\mathcal{X}$ can be obtained by gluing together $\mathcal{X}_{\mathrm{U}}$ and the complement $\mathcal{X}_{\backslash\mathrm{U}}$ along a "boundary" determined by U. The fracture theorem **(2·3)** also internalizes into the universes of the type theory, as pointed out by Rijke, Shulman, and Spitters [RSS20].

3 Realignment in synthetic Tait computability

(3·1) A syntactic metatheorem for a type theory $\mathbb{T}$ is proved by finding a suitable topos $\mathcal{X}$ equipped with an open immersion $j : \widehat{\mathbb{T}} \hookrightarrow \mathcal{X}$; the choice of $\mathcal{X}$ is determined by what sort of metatheorem one wishes to prove. Then, the main theorem is attacked by constructing an algebra for $\mathbb{T}$ over $\mathcal{X}$ that restricts along the open immersion to the syntactic algebra over $\widehat{\mathbb{T}}$.

(3·2) When constructing the "computability algebra" in **(3·1)**, we often end up with a construction that lies *not* over the corresponding sort in the syntactic algebra, but over one of its isomorphs. Consider, for instance, the case of the "walking type theory"; here, one has in $\mathscr{E} = \mathrm{Sh}(\mathcal{X})$ a $\bigcirc$-modal type $\mathsf{tp} : \mathscr{U}_{\bigcirc}$ and a family of $\bigcirc$-modal types $\mathsf{el} : \hat{\bigcirc}\mathsf{tp} \longrightarrow \mathscr{U}_{\bigcirc}$.

(3·3) For the computability algebra, we need to define a type $\mathsf{tp}^* : \mathscr{U}$ such that $\forall z : [\mathrm{U}].\mathsf{tp}^* = \mathsf{tp}(z)$ strictly. Our first attempt will satisfy this condition only up to isomorphism:

$$\mathsf{tp}^* = \textstyle\sum_{\mathrm{A}:\hat{\bigcirc}\mathsf{tp}}\{\mathrm{A}^* : \mathscr{U} \mid \forall z : \mathrm{U}.\mathrm{A}^* = \mathsf{el}(\mathrm{A})(z)\} \qquad (*)$$

We obviously have $\forall z : [\mathrm{U}].\mathsf{tp}^* \cong \mathsf{tp}(z)$ because under the assumption of a proof of [U], the second component of the dependent sum is a singleton. If the Orton–Pitts strictification axiom **(0·1)** holds for the universe $\mathscr{U}$, we can *realign* our construction to extend tp strictly as desired, choosing $\phi := \mathrm{U}$. The realignment principle is used again and again by Sterling and Harper [SH20] to obtain simple and conceptual constructions of the computability structure for the connectives of type theory.

(3·4) Unfortunately, the Orton–Pitts axiom **(0·1)** is only known to hold for the Hofmann–Streicher universes in categories of presheaves; the sheafified universes à la Streicher [Str05] do not seem to have this property. Luckily, many gluing situations for dependent type theory end up involving only presheaf universes; but we will not always be so lucky, *e.g.* consider the the case of normalization of type theory with strict coproducts [Alt+01]. The purpose of this note is to give an alternative construction of universes on glued topoi that supports the realignment at a specific open $\mathrm{U} : \Omega$.

4 A construction of universes that support realignment

(4·1) In this section, we will work in the internal language of a sheaf topos $\mathcal{X}$ together with an open U; we assume the corresponding open and closed modalities, as well as their fracture theorem **(2·3)**. Let $\mathscr{U}$ refer to an ordinary universe in $\mathscr{E} = \mathrm{Sh}(\mathcal{X})$; we will treat the decodings of $\mathscr{U}$ implicitly, but it is important to remember that these decodings only commute with connectives up to isomorphism in a category of sheaves.

(4·2) We define a new universe $\mathsf{el}_{\mathscr{W}} : \mathscr{W} \to \mathscr{U}$ as follows, as in Rijke, Shulman, and Spitters [RSS20]:

$$
\begin{array}{l}
\textbf{record } \mathscr{W} \textbf{ where}\\
\quad \mathsf{opn} : \mathscr{U}_{\bigcirc}\\
\quad \mathsf{clsd} : \mathscr{U}_{\bullet}\\
\quad \mathsf{proj} : \hat{\bullet}\mathsf{clsd} \to \bullet\hat{\bigcirc}\mathsf{opn}
\end{array}
$$

$$
\begin{array}{l}
\textbf{record } \mathsf{el}_{\mathscr{W}}\ (\mathrm{A} : \mathscr{W}) \textbf{ where}\\
\quad \mathsf{opn} : \hat{\bigcirc}(\mathrm{A}.\mathsf{opn})\\
\quad \mathsf{clsd} : \hat{\bullet}(\mathrm{A}.\mathsf{clsd})\\
\quad \mathsf{match} : \mathrm{A}.\mathsf{proj}(\mathsf{clsd}) =_{\bullet\hat{\bigcirc}\mathrm{A}.\mathsf{opn}} \eta^{\bullet}_{\ldots}\mathsf{opn}
\end{array}
$$

It will *not* be the case that $\mathscr{W} \cong \mathscr{U}$; this is because $\mathscr{W}$ carries additional data that is not uniquely determined up to strict equality. However, this data *is* uniquely determined up to isomorphism by the fracture theorem **(2·3)**, which is the essential insight behind our construction.

(4·3) There is a "promotion" operator that takes a piece of the open part of some $\mathrm{A} : \mathscr{W}$ and turns it an entire element of A under the assumption of $z : [\mathrm{U}]$, recalling that $\bullet\mathrm{X} \cong \mathbf{1}$ in this context.

$$
\begin{aligned}
\Uparrow_z &: \mathrm{A}.\mathsf{opn}(z) \to \mathsf{el}_{\mathscr{W}}(\mathrm{A})\\
\Uparrow_z(a).\mathsf{opn} &= \eta^{\circ}_{\mathrm{A}.\mathsf{opn}(z)}(a)\\
\Uparrow_z(a).\mathsf{clsd} &= \star\\
\Uparrow_z(a).\mathsf{proj} &= \star
\end{aligned}
$$

(4·4) We may define a canonical "fracturing map" $\mathsf{frac} : \mathscr{U} \to \mathscr{W}$ as follows:

$$
\begin{aligned}
\mathsf{frac} &: \mathscr{U} \to \mathscr{W}\\
\mathsf{frac}(\mathrm{A}).\mathsf{opn} &= \eta^{\circ}_{\mathscr{U}}\mathrm{A}\\
\mathsf{frac}(\mathrm{A}).\mathsf{clsd} &= \eta^{\bullet}_{\mathscr{U}}\mathrm{A}\\
\mathsf{frac}(\mathrm{A}).\mathsf{proj}(m) &= x \leftarrow m\,;\, \eta^{\bullet}_{\bigcirc\mathrm{A}}\eta^{\circ}_{\mathrm{A}}x
\end{aligned}
$$

For each $\mathrm{A} : \mathscr{U}$, we have an isomorphism $\mathsf{el}_{\mathscr{W}}(\mathsf{frac}(\mathrm{A})) \cong \mathrm{A}$; unfolding definitions, it is easy to see that this is *exactly* the content of the fracture theorem **(2·3)**. Hence, we see that $\mathscr{W}$ is generic for the same class of types as $\mathscr{U}$.

(4·5) *Realignment.* Let $\mathrm{A} : \mathscr{W}$ and $\mathrm{A}_{\mathrm{U}} : \mathscr{U}_{\bigcirc}$ together with a modal function $f_{\mathrm{U}} : \hat{\bigcirc}(\lambda z.\mathsf{el}_{\mathscr{W}}(\mathrm{A}) \to \mathrm{A}_{\mathrm{U}}(z))$. We can *realign* A to lie over A_{U} as follows:

$$
\begin{aligned}
\mathsf{Glue}(\mathrm{A},\mathrm{A}_{\mathrm{U}},f_{\mathrm{U}}).\mathsf{opn} &= \mathrm{A}_{\mathrm{U}}\\
\mathsf{Glue}(\mathrm{A},\mathrm{A}_{\mathrm{U}},f_{\mathrm{U}}).\mathsf{clsd} &= \mathrm{A}.\mathsf{clsd}\\
\mathsf{Glue}(\mathrm{A},\mathrm{A}_{\mathrm{U}},f_{\mathrm{U}}).\mathsf{proj}(m) &= a \leftarrow \mathrm{A}.\mathsf{proj}(m)\,;\, \eta^{\bullet}_{\ldots}(\lambda z.f_{\mathrm{U}}(z, \Uparrow_z a(z)))
\end{aligned}
$$

We additionally have an introduction map $\mathsf{glue}_{\mathrm{A},\mathrm{A}_{\mathrm{U}},f_{\mathrm{U}}}$:

$$
\mathsf{glue}_{\mathrm{A},\mathrm{A}_{\mathrm{U}},f_{\mathrm{U}}} : \mathsf{el}_{\mathscr{W}}(\mathrm{A}) \to \mathsf{el}_{\mathscr{W}}(\mathsf{Glue}(\mathrm{A},\mathrm{A}_{\mathrm{U}},f_{\mathrm{U}}))
$$

$$\begin{aligned}\mathsf{glue}_{\mathrm{A},\mathrm{A}_\mathrm{U},f_\mathrm{U}}(x).\mathsf{opn} &= \lambda z.f_\mathrm{U}(z,x)\\ \mathsf{glue}_{\mathrm{A},\mathrm{A}_\mathrm{U},f_\mathrm{U}}(x).\mathsf{clsd} &= x.\mathsf{clsd}\\ \mathsf{glue}_{\mathrm{A},\mathrm{A}_\mathrm{U},f_\mathrm{U}}(x).\mathsf{match} &= \star\end{aligned}$$

To substantiate the last clause $\mathsf{glue}_{\mathrm{A},\mathrm{A}_\mathrm{U},f_\mathrm{U}}(x).\mathsf{match}$, we must check the following equation:

$$(\eta^{\bullet}_{\dots}(\lambda z.f_\mathrm{U}(z,x))) = (a \leftarrow \mathrm{A}.\mathsf{proj}(x.\mathsf{clsd})\ ;\ \eta^{\bullet}_{\dots}(\lambda z.f_\mathrm{U}(z,\Uparrow_z a(z)))) \tag{4·5·1}$$

From $x.\mathsf{match}$, we have $\mathrm{A}.\mathsf{proj}(x.\mathsf{clsd}) = \eta^{\bullet}_{\dots}(x.\mathsf{opn})$, hence we may rewrite our goal:

$$(\eta^{\bullet}_{\dots}(\lambda z.f_\mathrm{U}(z,x))) = (\eta^{\bullet}_{\dots}(\lambda z.f_\mathrm{U}(z,\Uparrow_z(x.\mathsf{opn}(z))))) \tag{4·5·2}$$

Under the assumption $z : [\mathrm{U}]$, we have $\Uparrow_z(x.\mathsf{opn}(z)) = x$, however, so we are done.

(4·6) Under the assumptions of **(4·5)**, further assume that f_U is a modal isomorphism with modal inverse $\bar{f}_\mathrm{U}$. We may define an elimination map unglue that we will later see to be an inverse to $\mathsf{glue}_{\mathrm{A},\mathrm{A}_\mathrm{U},f_\mathrm{U}}$:

$$\begin{aligned}\mathsf{unglue}_{\mathrm{A},\mathrm{A}_\mathrm{U},\bar{f}_\mathrm{U}} &: \mathsf{el}_{\mathscr{W}}(\mathsf{Glue}(\mathrm{A},\mathrm{A}_\mathrm{U},f_\mathrm{U})) \to \mathsf{el}_{\mathscr{W}}(\mathrm{A})\\ \mathsf{unglue}_{\mathrm{A},\mathrm{A}_\mathrm{U},\bar{f}_\mathrm{U}}(x).\mathsf{opn} &= \lambda z.(\bar{f}_\mathrm{U}(z,x.\mathsf{opn}(z))).\mathsf{opn}(z)\\ \mathsf{unglue}_{\mathrm{A},\mathrm{A}_\mathrm{U},\bar{f}_\mathrm{U}}(x).\mathsf{clsd} &= x.\mathsf{clsd}\\ \mathsf{unglue}_{\mathrm{A},\mathrm{A}_\mathrm{U},\bar{f}_\mathrm{U}}(x).\mathsf{match} &= \star\end{aligned}$$

For the last clause $\mathsf{unglue}_{\mathrm{A},\mathrm{A}_\mathrm{U},\bar{f}_\mathrm{U}}(x).\mathsf{match}$, we must check the following equation:

$$\mathrm{A}.\mathsf{proj}(x.\mathsf{clsd}) = \eta^{\bullet}_{\dots}(\lambda z.g(z,x.\mathsf{opn}(z)).\mathsf{opn}(z)) \tag{4·6·1}$$

From $x.\mathsf{match}$, we have the following:

$$\big(a \leftarrow \mathrm{A}.\mathsf{proj}(x.\mathsf{clsd})\ ;\ \eta^{\bullet}_{\dots}(\lambda z.f_\mathrm{U}(z,\Uparrow_z a(z)))\big) = (\eta^{\bullet}_{\dots}\,x.\mathsf{opn}) \tag{4·6·2}$$

Keeping Equation (4·6·2) in mind, we rewrite the right-hand side of Equation (4·6·1):

$$\begin{aligned}&\eta^{\bullet}_{\dots}(\lambda z.\bar{f}_\mathrm{U}(z,x.\mathsf{opn}(z)).\mathsf{opn}(z))\\ &\quad= u \leftarrow \eta^{\bullet}_{\dots}\,x.\mathsf{opn}\ ;\ \eta^{\bullet}_{\dots}(\lambda z.\bar{f}_\mathrm{U}(z,u(z)).\mathsf{opn}(z))\\ &\quad= u \leftarrow \big(a \leftarrow \mathrm{A}.\mathsf{proj}(x.\mathsf{clsd})\ ;\ \eta^{\bullet}_{\dots}(\lambda z.f_\mathrm{U}(z,\Uparrow_z a(z)))\big)\ ;\ \eta^{\bullet}_{\dots}(\lambda z.\bar{f}_\mathrm{U}(z,u(z)).\mathsf{opn}(z))\\ &\quad= a \leftarrow \mathrm{A}.\mathsf{proj}(x.\mathsf{clsd})\ ;\ \eta^{\bullet}_{\dots}(\lambda z.\bar{f}_\mathrm{U}(z, f_\mathrm{U}(z,\Uparrow_z a(z))).\mathsf{opn}(z))\\ &\quad= a \leftarrow \mathrm{A}.\mathsf{proj}(x.\mathsf{clsd})\ ;\ \eta^{\bullet}_{\dots}(\lambda z.\Uparrow_z a(z).\mathsf{opn}(z))\\ &\quad= a \leftarrow \mathrm{A}.\mathsf{proj}(x.\mathsf{clsd})\ ;\ \eta^{\bullet}_{\dots}(a)\\ &\quad= \mathrm{A}.\mathsf{proj}(x.\mathsf{clsd})\end{aligned}$$

(4·7) Under the assumptions of **(4·5)** and **(4·6)**, we observe that $\mathsf{unglue}_{\mathrm{A},\mathrm{A}_\mathrm{U},\bar{f}_\mathrm{U}}$ is both left and right inverse to $\mathsf{glue}_{\mathrm{A},\mathrm{A}_\mathrm{U},f_\mathrm{U}}$; on $x.\mathsf{opn}$, this follows from the fact that f_U is inverse to $\bar{f}_\mathrm{U}$, and on the closed part, it follows because neither function does anything to $x.\mathsf{clsd}$.

5 Concluding remarks

(5·1) Our strictification construction apparently suffices for the needs of synthetic Tait computability, where one is realigning always at a *single* proposition $\mathrm{U} : \Omega$. It does not appear to be enough, however, to validate the full Orton–Pitts strictification axiom **(0·1)**, so it remains an open question whether the methods of Orton and Pitts [OP16] can be used to construct models of univalent type theory in non–presheaf topoi.

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