Guarded Computational Type Theory

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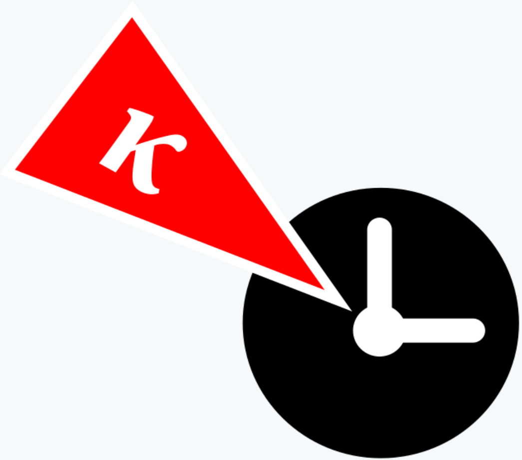

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Variety of techniques employed today, but the problem of mixed variance fixed points remained the fundamental struggle of PL semantics.

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$$\llbracket A \to B \rrbracket\rho \triangleq \{\lambda x.e \mid \forall v \in \llbracket A \rrbracket\rho.\ e[v/x] \in \llbracket B \rrbracket\rho\}$$

(no!! not well-defined)

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Using ideas from Nakano [2000], abstract version of step-indexing factored in terms of *approximation modality* $\triangleright$, called “later” [Appel et al., 2007].

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not just for predicates! can also form guarded-recursive "sets":

$$\mathrm{gstream} \cong \mathbb{N} \times \triangleright\mathrm{gstream}$$

Programming Example: Guarded Streams

```
type gstr[X] = cons of X * |> gstr[X]

let head (xs : gstr[X]) : X =
  let cons (x, _) = xs in x

let tail (xs : gstr[X]) : |> gstr[X] =
  let cons (_, ys) = xs in ys

let zipWith (f : X -> Y -> Z) : gstr[X] -> gstr[Y] -> gstr[Z] =
  gfix F in
    fun (cons (x, xs)) (cons (y, ys)) ->
      cons (f x y, F <*> xs <*> ys)
```

Can't write unguarded `tail` function! Guarded-recursive types ensure that all observations are *causal*.

Constant modality

Models of guarded recursion often include another modal operator $\square$ which *neutralizes* $\triangleright$:

$$\square A \to A \qquad \square A \to \square\square A \qquad \square \triangleright A \to \square A$$

Converts guarded recursion to coinduction.

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$$\begin{aligned} \text{gstream} &\cong \mathbb{N} \times \triangleright \text{gstream} \\ \text{sequence} &\cong \mathbb{N} \times \text{sequence} \end{aligned}$$

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$$\mathsf{gstream}_\kappa \cong \mathbb{N} \times \triangleright_\kappa \mathsf{gstream}_\kappa$$
$$\forall\kappa.\mathsf{gstream}_\kappa \cong \mathbb{N} \times \forall\kappa.\mathsf{gstream}_\kappa$$

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Most advances in the area of guarded higher-order logics, but **dependent type theory** indispensible.

Dependent type theory = semantic framework to unify the study of *programs* with the study of *programming languages.*

Guarded Dependent Type Theory

Bizjak and Møgelberg [2017] present elegant denotational account of guarded dependent type theory ($\Pi, \Sigma, \triangleright_\kappa, \forall\kappa, \ldots$)

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1. what about operational meaning? (status unknown for GDTT)

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A couple things gave us pause...

1. what about operational meaning? (status unknown for GDTT)
2. universes $\mathcal{U}_i$ are essential, but not directly supported in GDTT semantics

What's the deal with universes?

Adequacy of $\forall\kappa$ to encode coinduction requires that functions of clocks are constant. Called **clock irrelevance**.

But what about $(\lambda\kappa.\lambda A.\ \triangleright_\kappa A) \in \forall\kappa.(\mathcal{U} \to \mathcal{U})$?

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global orthogonality constraint

Idea: get rid of ordinary universes, replace with weaker clock-context-indexed universes $\mathcal{U}_i^{\vec{\kappa}}$.

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We call it Guarded Computational Type Theory / CTT🕒.

Solution to universe problem

Inspired by extension of propositions-as-types discipline to *intersection* connective [Bickford and Constable, 2012, Allen et al., 2006].

Decompose clock quantifiers into two parts:

1. *uniform* quantifier (intersection): $\{\kappa \div \mathtt{clk}\} \to A_\kappa$
2. *non-uniform* quantifier (product): $(\kappa : \mathtt{clk}) \to A_\kappa$

Idea: don't impose global orthogonality condition, instead change the quantifier.

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Programs have more than one behavior: in particular, the specification $M \in \{\kappa \div \texttt{clk}\} \to A_\kappa$ means that M exhibits *all* the behaviors A_κ simultaneously.

"Who among us…?"

Initial version of **CTT**⌚ had subtle+critical bug, wasn't clear if it could be salvaged. Need for formalization.

Formalizing CTT$^{\circledcirc}$

To gain confidence, formalized corrected version in **Coq** using internal topos-theoretic technique.

Basic idea: design a suitable presheaf topos $\mathcal{S}_{\circledcirc}$ whose internal logic has clocks, $\triangleright_\kappa$ and $\forall\kappa$. See our paper, but think of presheaves on finitely many copies of ω.

The logic of clocks

There is a *presheaf of clocks* $\mathbb{K} : \mathcal{S}_{\circlearrowleft}$.

The topos $\mathcal{S}_{\circlearrowleft}$ contains a rich higher-order logic with a $\mathbb{K}$-indexed family of modalities $\triangleright_\kappa$, enjoying crucial principles including the following:

IRRELEVANCE
$\exists\kappa : \mathbb{K}.\, \top$

"IMPATIENCE"
$\forall\phi : \Omega^{\mathbb{K}}.\, \left(\forall\kappa : \mathbb{K}.\, \triangleright_\kappa \phi(\kappa)\right) \Rightarrow \forall\kappa : \mathbb{K}.\, \phi(\kappa)$

MONOTONICITY
$\forall\kappa : \mathbb{K}.\, \forall\phi : \Omega.\, \phi \Rightarrow \triangleright_\kappa \phi$

$\wedge$-DISTRIBUTIVITY
$\forall\kappa : \mathbb{K}.\, \forall\phi, \psi : \Omega.\, \triangleright_\kappa(\phi \wedge \psi) \equiv \left(\triangleright_\kappa \phi \wedge \triangleright_\kappa \psi\right)$

$\Rightarrow$-DISTRIBUTIVITY
$\forall\kappa : \mathbb{K}.\, \forall\phi, \psi : \Omega.\, \triangleright_\kappa(\phi \Rightarrow \psi) \equiv \left(\triangleright_\kappa \phi \Rightarrow \triangleright_\kappa \psi\right)$

STRONG LÖB INDUCTION
$\forall\kappa : \mathbb{K}.\, \forall\phi : \Omega.\, \left(\triangleright_\kappa \phi \Rightarrow \phi\right) \Rightarrow \phi$

Internal Syntax and Operational Semantics

Key idea: higher-order internal encoding of syntax for later modality and clock quantification[1], De Bruijn encoding for ordinary binders:

$$\frac{\kappa : \mathbb{K} \qquad A : \mathcal{P}rog}{\blacktriangleright_{\kappa} A : \mathcal{P}rog}$$

$$\frac{A : \mathcal{P}rog^{\mathbb{K}}}{(\kappa : \mathtt{clk}) \to A(\kappa) : \mathcal{P}rog} \qquad \frac{A : \mathcal{P}rog^{\mathbb{K}}}{\{\kappa \div \mathtt{clk}\} \to A(\kappa) : \mathcal{P}rog}$$

[1] See Fiore et al. [1999], Hofmann [1999].

Next, we construct the PER-model of type theory inside the *internal logic* of $\mathcal{S}_{\circlearrowleft}$.

Idea: internal inductive-recursive definition of a universe hierarchy, assigning relations to type-codes.

All clauses are standard, except the relations which define the clock-related connectives; approximately:

$$
\begin{aligned}
[\![\blacktriangleright_\kappa A]\!] &= \{(M, N) \mid \triangleright_\kappa((M, N) \in [\![A]\!])\} \\
[\![(\kappa : \mathtt{clk}) \to A(\kappa)]\!] &= \{(M, N) \mid \forall \kappa : \mathbb{K}.\, (\mathtt{app}(M, \kappa), \mathtt{app}(N, \kappa)) \in [\![A(\kappa)]\!]\} \\
[\![\{\kappa \div \mathtt{clk}\} \to A(\kappa)]\!] &= \{(M, N) \mid \forall \kappa : \mathbb{K}.\, (M, N) \in [\![A(\kappa)]\!]\} \\
&\vdots
\end{aligned}
$$

Programming in CTT🕒

Using a combination of both clock quantifiers and the later modality, we can program the type of guarded-recursive streams “qua fixed points on universes”.[2]

$$\mathsf{gstr} \in (\_ : \mathtt{clk}) \to \mathtt{U}_0 \to \mathtt{U}_0$$
$$\mathsf{gstr} = \lambda\kappa.\,\mathtt{fix}\ F\ \mathtt{in}\ \lambda A.\, A \times \blacktriangleright_\kappa F(A)$$

$$\mathsf{seq} \in \mathtt{U}_0 \to \mathtt{U}_0$$
$$\mathsf{seq} = \lambda A.\,\{\kappa \div \mathtt{clk}\} \to \mathsf{gstr}\ \kappa\ A$$

$$\mathsf{gzipwith} \in (f : X \to Y \to Z) \to \{\kappa \div \mathtt{clk}\} \to \mathsf{gstr}\ \kappa\ X \to \mathsf{gstr}\ \kappa\ Y \to \mathsf{gstr}\ \kappa\ Z$$
$$\mathsf{gzipwith} = \lambda f.\,\mathtt{fix}\ F\ \mathtt{in}\ \lambda\alpha,\beta.\,\langle f\ \alpha.1\ \beta.1, F\ \alpha.2\ \beta.2\rangle$$

$$\mathsf{zipwith} \in (f : X \to Y \to Z) \to \mathsf{seq}\ X \to \mathsf{seq}\ Y \to \mathsf{seq}\ Z$$
$$\mathsf{zipwith} = \mathsf{gzipwith}$$

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[2] Birkedal and Møgelberg [2013]

Summary of contribution

In **Guarded Computational Type Theory** / **CTT⌚**, we propose a guarded dependent type theory which supports a predicative hierarchy of standard/*non-indexed* type-theoretic universes $\mathtt{U}_i$.

1. standard dependent type connectives (Π,Σ, $\mathtt{Eq}$)
2. clock-indexed later modality ($\blacktriangleright_\kappa A$)
3. parametric clock quantifier ($\{\kappa \div \mathtt{clk}\} \to A$)
4. non-parametric clock quantifier ($(\kappa : \mathtt{clk}) \to A$)
5. type-theoretic universes $\mathtt{U}_i$ which contain $\blacktriangleright_\kappa A$
6. operational semantics, canonicity result
7. **formalized in Coq using internal-topos-theoretic technique**

(questions?)

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