

Normalization by gluing for free λ -theories

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Abstract

The connection between normalization by evaluation, logical predicates and semantic gluing constructions is a matter of folklore, worked out in varying degrees within the literature. In this note, we present an elementary version of the gluing technique which corresponds closely with both semantic normalization proofs and the syntactic normalization by evaluation

We will expand in more detail the insight presented in Streicher’s short note [Streicher, 1998] and in Fiore [2002], giving some explicit constructions. We will be considering the case of free λ -theories generated from many-typed first-order signatures.

1 λ -signatures and λ -theories

Definition 1.1 (Arity). A simply-typed first-order arity for a set T of atomic types is a pair $\alpha \equiv (\vec{\sigma}, \tau)$ of a list of types and a types. We write $T^\dagger \stackrel{\text{def}}{=} T^* \times T$ for the set of such arities.

Definition 1.2 (Many-typed signature). Following Jacobs [1999], a *many-typed signature* $\Sigma \equiv (\mathcal{U}, \partial)$ is a set of atomic types \mathcal{U} together with an arity-indexed family of sets of operations ∂ , taking each operation ϑ to $\partial(\vartheta) \in \mathcal{U}^\dagger$. More abstractly, the category \mathfrak{Sig} of such signatures arises as the pullback of the fundamental fibration along the arity endofunctor.

$$\begin{array}{ccc} \mathfrak{Sig} & \xrightarrow{\partial} & \mathfrak{Set}^{\rightarrow} \\ \mathcal{U} \downarrow \lrcorner & & \downarrow \text{cod} \\ \mathfrak{Set} & \xrightarrow{(-)^\dagger} & \mathfrak{Set} \end{array}$$

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From a collection of atomic types \mathcal{U} , we generate the type structure of the λ -calculus as the least set $\overline{\mathcal{U}}$ closed under the following formation rules:

$$\frac{\tau \in \mathcal{U}}{\tau \in \overline{\mathcal{U}}} \quad \frac{\sigma \in \overline{\mathcal{U}} \quad \tau \in \overline{\mathcal{U}}}{\sigma \times \tau \in \overline{\mathcal{U}}} \quad \frac{\sigma \in \overline{\mathcal{U}} \quad \tau \in \overline{\mathcal{U}}}{\sigma \rightarrow \tau \in \overline{\mathcal{U}}}$$

1.1 The clone of a λ -signature

From a λ -signature Σ we can freely generate a special family of sets $\text{Cn}_\Sigma(\Gamma, \tau)$ called its *clone*, indexed in $(\Gamma, \tau) \in \overline{\mathcal{U}}_\Sigma^\dagger$; simultaneously, we define the indexed set of substitutions $\text{Sb}_\Sigma(\Gamma, \Delta)$, indexed in $(\Gamma, \Delta) \in \overline{\mathcal{U}}_\Sigma^* \times \overline{\mathcal{U}}_\Sigma^*$. In our presentation, we choose to use *explicit substitutions* rather than implicit substitutions, because they are more abstract and allow a formulation without explicit reference to preterms. Additionally, explicit substitutions scale up to the metatheory of dependent type theory in a way that the implicit (admissible) notion of substitution on preterms cannot.

We will write $\Gamma \vdash_\Sigma t : \tau$ and $\Gamma \vdash_\Sigma \delta : \Delta$ to mean that $t \in \text{Cn}_\Sigma(\Gamma, \tau)$ and $\delta \in \text{Sb}_\Sigma(\Gamma, \Delta)$ respectively, and $\Gamma \vdash_\Sigma s = t : \tau$ and $\Gamma \vdash_\Sigma \delta_0 = \delta_1 : \Delta$ to mean that s and t are equal elements of $\text{Cn}_\Sigma(\Gamma, \tau)$ and δ_0 and δ_1 are equal elements of $\text{Sb}_\Sigma(\Gamma, \Delta)$ respectively. In our notation, we *presuppose* that s and t are elements of the clone when we state that they are equal.

The clone of Σ is defined as a quotient using the following indexed inductive definition:

$$\begin{array}{c} \text{VARIABLE} \\ \Gamma, \tau \vdash_\Sigma v : \tau \\ \\ \text{OPERATION} \\ \frac{\partial_\Sigma(\vartheta) \equiv (\Gamma, \tau)}{\Gamma \vdash_\Sigma \vartheta : \tau} \\ \\ \text{SUBST} \\ \frac{\Gamma \vdash_\Sigma \delta : \Delta \quad \Delta \vdash_\Sigma t : \tau}{\Gamma \vdash_\Sigma t[\delta] : \tau} \\ \\ \text{ABSTRACTION} \\ \frac{\Gamma, \sigma \vdash_\Sigma t : \tau}{\Gamma \vdash_\Sigma \lambda^\sigma(t) : \sigma \rightarrow \tau} \\ \\ \text{APPLICATION} \\ \frac{\Gamma \vdash_\Sigma s : \sigma \rightarrow \tau \quad \Gamma \vdash_\Sigma t : \sigma}{\Gamma \vdash_\Sigma s(t) : \tau} \\ \\ \text{PAIR} \\ \frac{\Gamma \vdash_\Sigma s : \sigma \quad \Gamma \vdash_\Sigma t : \tau}{\Gamma \vdash_\Sigma (s, t) : \sigma \times \tau} \\ \\ \text{PROJ1} \\ \frac{\Gamma \vdash_\Sigma s : \sigma \times \tau}{\Gamma \vdash_\Sigma s.1 : \sigma} \\ \\ \text{PROJ2} \\ \frac{\Gamma \vdash_\Sigma s : \sigma \times \tau}{\Gamma \vdash_\Sigma s.2 : \tau} \\ \\ \text{SB/IDN} \\ \Gamma \vdash_\Sigma \text{id}_\Gamma : \Gamma \\ \\ \text{SB/PROJ} \\ \Gamma, \tau \vdash_\Sigma p : \Gamma \\ \\ \text{SB/EXT} \\ \frac{\Gamma \vdash_\Sigma \delta : \Delta \quad \Gamma \vdash_\Sigma t : \tau}{\Gamma \vdash_\Sigma \delta.t : \Delta, \tau} \\ \\ \text{SB/CMP} \\ \frac{\Gamma \vdash_\Sigma \delta : \Delta \quad \Delta \vdash_\Sigma \xi : \Xi}{\Gamma \vdash_\Sigma \xi \circ \delta : \Xi} \end{array}$$

Before we define the equivalence relation by which the clone is quotiented, it will be useful to define an auxiliary substitution for De Bruijn lifting:

$$\uparrow[\gamma] \stackrel{\text{def}}{=} (p \circ \gamma).v$$

Next, we generate equivalence relations on terms and substitutions from the following rules:

$$\begin{array}{c}
\text{APP/BETA} \qquad \qquad \qquad \text{ABS/ETA} \\
\Gamma \vdash_{\Sigma} (\lambda^{\sigma}(t))(s) = t[\text{id}_{\Gamma}.s] : \tau \qquad \qquad \Gamma \vdash_{\Sigma} t = \lambda^{\sigma}((t[p])(v)) : \sigma \rightarrow \tau \\
\\
\text{FST/BETA} \qquad \qquad \text{SND/BETA} \qquad \qquad \text{PAIR/ETA} \\
\Gamma \vdash_{\Sigma} (s, t).1 = s : \sigma \qquad \Gamma \vdash_{\Sigma} (s, t).2 = t : \tau \qquad \Gamma \vdash_{\Sigma} t = (t.1, t.2) : \sigma \times \tau \\
\\
\text{SB/CMP/IDN/L} \qquad \qquad \text{SB/CMP/IDN/R} \qquad \qquad \text{SB/CMP/ASSOC} \\
\Gamma \vdash_{\Sigma} \text{id}_{\Delta} \circ \delta = \delta : \Delta \qquad \Gamma \vdash_{\Sigma} \delta \circ \text{id}_{\Gamma} = \delta : \Delta \qquad \Gamma \vdash_{\Sigma} \gamma \circ (\delta \circ \xi) = (\gamma \circ \delta) \circ \xi : \Delta \\
\\
\text{SB/CMP/PROJ} \qquad \qquad \text{SB/CMP/DOT} \qquad \qquad \text{SB/VAR/IDN} \\
\Gamma \vdash_{\Sigma} p \circ \delta.t = \delta : \Delta \qquad \Gamma \vdash_{\Sigma} \delta \circ (\xi.t) = (\delta \circ \xi).t[\delta] : \Delta \qquad \Gamma \vdash_{\Sigma} v[\text{id}_{\Gamma}] = v : \tau \\
\\
\text{SB/VAR/EXT} \qquad \qquad \text{SB/ABS} \\
\Gamma \vdash_{\Sigma} v[\delta.t] = t : \tau \qquad \Gamma \vdash_{\Sigma} (\lambda^{\sigma}(t))[\delta] = \lambda^{\sigma}(t[\uparrow[\delta]]) : \sigma \rightarrow \tau \\
\\
\text{SB/APP} \qquad \qquad \qquad \text{SB/PAIR} \\
\Gamma \vdash_{\Sigma} (t(s))[\delta] = t[\delta](s[\delta]) : \tau \qquad \Gamma \vdash_{\Sigma} (s, t)[\delta] = (s[\delta], t[\delta]) : \sigma \times \tau \\
\\
\text{SB/PROJ1} \qquad \qquad \qquad \text{SB/PROJ2} \\
\Gamma \vdash_{\Sigma} t.1[\delta] = (t[\delta]).1 : \sigma \qquad \Gamma \vdash_{\Sigma} t.2[\delta] = (t[\delta]).2 : \tau
\end{array}$$

We omit the congruence cases for brevity. The clone of Σ is now defined as the indexed family of quotients generated by the formation and definitional equivalence rules given above.

Representation of the quotient In these notes, we will not dwell on the technical representation of the quotiented terms. However, we will remark that the most convenient induction principle for the quotiented syntax would arrive from a presentation as a *quotient inductive type*; moreover, our inductive definition falls under a schema for *finitary* quotient inductive types which is already known to be interpretable in the setoid model of type theory [Dybjer and Moeneclaey, 2018].

1.2 The classifying category of a λ -signature

We can see that the language of substitutions above has the structure of category; this category is in fact called Cl_{Σ} , the *classifying category* or *Lawvere category* of the λ -signature Σ . The classifying category is also just called the (pure) λ -theory generated by the signature. Concretely, it has contexts Γ as objects, and equivalence classes of substitutions $\Gamma \vdash_{\Sigma} \delta : \Delta$ as morphisms.

Proposition 1.3. *The classifying category Cl_{Σ} is cartesian closed.*

1.3 The category of renamings

Every λ -signature gives rises to another category, namely the *category of renamings* Ren_{Σ} . Abstractly, this can be characterized as the free strictly associative cartesian category

generated by $\widetilde{\mathcal{U}}_\Sigma$; concretely, its objects are contexts Γ , and its morphisms $\psi : \text{Ren}_\Sigma[\Gamma, \Delta]$ are vectors of projections (indices) from Γ into the types in Δ .

An explicit presentation of Ren_Σ appears in Fiore [2002, 2005] as the opposite of the comma category $[-] \downarrow \text{Const}(\widetilde{\mathcal{U}}_\Sigma)$, where $[-] : \mathbb{F} \rightarrow \mathbf{Set}$ takes a finite cardinal to a set. Fiore writes $\mathbb{F} \downarrow \widetilde{\mathcal{U}}_\Sigma$ for this comma construction, and $\mathbb{F}[\widetilde{\mathcal{U}}_\Sigma]$ for its opposite. Another possible presentation of Ren_Σ is as the subcategory of Cl_Σ which has the same objects, but whose morphisms are all of the form id_Γ or $\text{id}_\Gamma.t_0 \cdots t_n$ for some $n > 0$, with t_i of the form $\nu[\mathfrak{p}^k]$, writing \mathfrak{p}^k for the k -fold composition of \mathfrak{p} with itself.

2 Normalization and the Yoneda embedding

Working in a constructive metatheory, we can see that the intensional content of a certain natural isomorphism hides within it a normalization function for the lambda calculus over Σ , as observed in Čubrić et al. [1998]. Let $\widehat{\text{Cl}}_\Sigma$ denote the category of presheaves over the classifying category of Σ .

The Yoneda embedding is a cartesian closed functor $\mathfrak{y} : \text{Cl}_\Sigma \rightarrow \widehat{\text{Cl}}_\Sigma$, defined as $\mathfrak{y}\Delta = \text{Cl}_\Sigma[-, \Delta]$. Within the presheaf topos, it is easiest to think of the representable objects $\mathfrak{y}\Delta$ as the “type of substitutions into Δ ”.

There is another way to define the Yoneda embedding, which we will see is naturally isomorphic to what is written above. In this version, we define a functor $[\mathfrak{y}] : \text{Cl}_\Sigma \rightarrow \widehat{\text{Cl}}_\Sigma$ by recursion on the objects of Cl_Σ . For an atomic type $\tau \in \mathcal{U}_\Sigma$, $[\mathfrak{y}]\tau = \mathfrak{y}\tau$; but the remainder of the cases are defined using the cartesian closed structure of the presheaf topos instead of the cartesian closed structure of the classifying category:

$$\begin{aligned} [\mathfrak{y}](\sigma \times \tau) &= [\mathfrak{y}]\sigma \times [\mathfrak{y}]\tau \\ [\mathfrak{y}](\sigma \rightarrow \tau) &= [\mathfrak{y}]\tau^{[\mathfrak{y}]\sigma} \end{aligned}$$

Now, because the Yoneda embedding is cartesian closed, it is easy to see that we have a natural isomorphism $\mathfrak{y} \cong [\mathfrak{y}]$. However, observe that the elements in the fibers of $[\mathfrak{y}]$ are not λ -terms, but a kind of eta-long Boehm-tree representation of λ -terms.

That is, whereas the action of \mathfrak{y} on a syntactic morphism/term $\Gamma \vdash_\Sigma t : \sigma \times \tau$ is to simply embed t into the appropriate presheaf fiber, the action of $[\mathfrak{y}]$ on the same term must take t to an element of $[\mathfrak{y}]\sigma \times [\mathfrak{y}]\tau$, that is, an actual *pair*. Considering the case where t is actually a variable, we can see that the action of these two embeddings is intensionally quite different. The other side of the natural isomorphism is witnessed by a “readback” operation, which reads one of these expanded Boehm trees into a syntactic term (which can be seen to be β -normal and η -long). The normalization operation obtained by composing these operations can be seen to be an instance of *normalization by evaluation*.

The problem with this kind of result, however, is that the categories have quotiented too much for us to be able to say in mathematical (rather than merely intuitive) language that we have obtained a normalization function. In particular, the normalization operation that we describe above is actually equal as a function to the identity. This is because the classifying category is already quotiented by definitional equivalence.

As summarized in Streicher [1998], there are two ways out of this situation. One is to use a higher-dimensional structure, such as partial equivalence relations or setoids, in order to structure the ambient category theory; then, in the spirit of Bishop’s constructive mathematics, we can observe the intension of the normalization operation at the same time as seeing that it is the identity in its extension. This approach was carried out in Čubrić et al. [1998] using P-category theory, a variant of E-category theory in which setoids are replaced by PERs.

Another more direct way is obtained from the *gluing construction* in category theory, where we will choose a different semantic domain which allows us to see the difference between the two ways of interpreting syntax into the presheaf category. This was carried out in detail in Altenkirch et al. [1995], but in a manner that is unfortunately different enough from the classical construction that it is unclear how it relates. In Fiore [2002], normalization by evaluation for typed lambda calculus is related explicitly to gluing; what we present in these notes can be seen as an explicit instantiation of Fiore’s framework.

3 Normalization by gluing

To resolve the problem described above in Section 2, we will work with a more refined base category, namely the category of renamings Ren_Σ defined in Section 1.3. First observe that there is an inclusion of categories $i : \text{Ren}_\Sigma \rightarrow \text{Cl}_\Sigma$, since every context renaming can be represented as a substitution, a sequence of extensions by variables.

We have a reindexing functor $i^* : \widehat{\text{Cl}}_\Sigma \rightarrow \widehat{\text{Ren}}_\Sigma$ by precomposition. Composing with the Yoneda embedding, we can define a new functor $\mathfrak{E}m : \text{Cl}_\Sigma \rightarrow \widehat{\text{Ren}}_\Sigma$:

$$\begin{array}{ccc} \text{Cl}_\Sigma & & \\ \downarrow y & \searrow \mathfrak{E}m & \\ \widehat{\text{Cl}}_\Sigma & \xrightarrow{i^*} & \widehat{\text{Ren}}_\Sigma \end{array}$$

Relative hom functor As described in Fiore [2002], the functor $\mathfrak{E}m$ is called the “relative hom functor” of i , taking $\Delta : \text{Cl}_\Sigma$ to $\text{Cl}_\Sigma[i(-), \Delta]$. In Fiore [2002], this functor is written $\langle i \rangle$, whereas we write $\mathfrak{E}m$ in order to suggest the intuition that it defines a presheaf of open terms.

We have constructed $\mathfrak{E}m$ from the perspective of “adjusting” the Yoneda embedding from Cl_Σ , but Fiore [2002] explains another characterization of the same functor from the perspective of the Yoneda embedding from Ren_Σ . In particular, $\mathfrak{E}m$ is the left Kan extension of $y : \text{Ren}_\Sigma \rightarrow \widehat{\text{Ren}}_\Sigma$ along i :

$$\begin{array}{ccc} \text{Ren}_\Sigma & \xrightarrow{y} & \widehat{\text{Ren}}_\Sigma \\ \searrow i & \Downarrow & \nearrow \mathfrak{E}m \\ & \text{Cl}_\Sigma & \end{array}$$

3.1 Presheaves of neutrals and normals

In $\widehat{\text{Ren}}_\Sigma$, we can construct presheaves of neutral terms and normal terms for each type; note that such presheaves cannot be defined in $\widehat{\text{Cl}}_\Sigma$, because they crucially cannot be closed under arbitrary substitutions (whereas they happen to be closed under renamings). The fibers of these presheaves will have the property that the equality relation for their elements is *discrete*.

To be concrete, let us begin by defining some restricted typing judgments for neutrals and normals.

$$\begin{array}{c}
\text{VARIABLE} \\
\Gamma \vdash_\Sigma^{\text{ne}} \mathbf{v}[p^k] : \Gamma_{|\Gamma|-k-1} \\
\\
\text{OPERATION} \\
\frac{\partial_\Sigma(\vartheta) \equiv (\Delta, \tau) \quad \Gamma \vdash_\Sigma^{\text{nf}} \delta : \Delta}{\Gamma \vdash_\Sigma^{\text{ne}} \mathfrak{V}[\delta] : \tau} \\
\\
\text{APP} \quad \frac{\Gamma \vdash_\Sigma^{\text{ne}} t : \sigma \rightarrow \tau \quad \Gamma \vdash_\Sigma^{\text{nf}} s : \sigma}{\Gamma \vdash_\Sigma^{\text{ne}} t(s) : \tau} \quad \text{PROJ1} \quad \frac{\Gamma \vdash_\Sigma^{\text{ne}} t : \sigma \times \tau}{\Gamma \vdash_\Sigma^{\text{ne}} t.1 : \sigma} \quad \text{PROJ2} \quad \frac{\Gamma \vdash_\Sigma^{\text{ne}} t : \sigma \times \tau}{\Gamma \vdash_\Sigma^{\text{ne}} t.2 : \tau} \\
\\
\text{SHIFT} \quad \frac{\Gamma \vdash_\Sigma^{\text{ne}} t : \tau \quad \tau \in \mathcal{U}_\Sigma}{\Gamma \vdash_\Sigma^{\text{nf}} t : \tau} \quad \text{ABSTRACTION} \quad \frac{\Gamma, \sigma \vdash_\Sigma^{\text{nf}} t : \tau}{\Gamma \vdash_\Sigma^{\text{nf}} \lambda^\sigma(t) : \sigma \rightarrow \tau} \quad \text{PAIR} \quad \frac{\Gamma \vdash_\Sigma^{\text{nf}} s : \sigma \quad \Gamma \vdash_\Sigma^{\text{nf}} t : \tau}{\Gamma \vdash_\Sigma^{\text{nf}} (s, t) : \sigma \times \tau} \\
\\
\text{SB/NF/PROJ} \quad \Gamma \vdash_\Sigma^{\text{nf}} \mathbf{p}^{[\Gamma]} : [] \quad \text{SB/NF/EXT} \quad \frac{\Gamma \vdash_\Sigma^{\text{nf}} \delta : \Delta \quad \Gamma \vdash_\Sigma^{\text{nf}} t : \tau}{\Gamma \vdash_\Sigma^{\text{nf}} \delta.t : \Delta, \tau} \\
\\
\text{SB/NE/PROJ} \quad \Gamma \vdash_\Sigma^{\text{ne}} \mathbf{p}^{[\Gamma]} : [] \quad \text{SB/NE/EXT} \quad \frac{\Gamma \vdash_\Sigma^{\text{ne}} \delta : \Delta \quad \Gamma \vdash_\Sigma^{\text{ne}} t : \tau}{\Gamma \vdash_\Sigma^{\text{ne}} \delta.t : \Delta, \tau}
\end{array}$$

Admissible substitutions We have restricted the language of normal substitutions to consist in vectors of terms, constructed using the SB/PROJ and SB/EXT rules. The identity substitution is *admissible* as a neutral substitution, but is not one of the generators. We define $\Gamma \vdash_\Sigma^{\text{ne}} \text{id}_\Gamma : \Gamma$ by recursion on Γ as follows:

$$\begin{aligned}
\text{id}_{[]} &= p^0 \\
\text{id}_{\Gamma, \tau} &= \text{id}_\Gamma.v[p^0]
\end{aligned}$$

η -long normal forms Observe that we have ensured an η -long normal form by restricting the shift rule above to apply only at atomic types. It is easy to see that these judgments are closed under context renamings, i.e. support a Ren_Σ -action. Therefore, we can use these judgments as the raw material from which to build the presheaves of neutrals and

normals for each type $\tau \in \widehat{\mathcal{U}}_\Sigma$ as follows:

$$\begin{aligned} \mathfrak{Ne}_\tau &: \widehat{\text{Ren}}_\Sigma \\ \mathfrak{Ne}_\tau(\Gamma) &\equiv \{t \mid \Gamma \vdash_\Sigma^{\text{ne}} t : \tau\} \\ \mathfrak{Nf}_\tau &: \widehat{\text{Ren}}_\Sigma \\ \mathfrak{Nf}_\tau(\Gamma) &\equiv \{t \mid \Gamma \vdash_\Sigma^{\text{nf}} t : \tau\} \end{aligned}$$

3.2 Syntax with binding, internally

So far we have developed three presheaves of syntax in $\widehat{\text{Ren}}_\Sigma$: the presheaf of typed terms $\mathfrak{Sm}(\tau)$, the presheaf of neutrals \mathfrak{Ne}_τ and the presheaf of normals \mathfrak{Nf}_τ . Using the internal language of the functor category, we can justify a simpler “higher-order” notation for working with elements of these presheaves internally [Hofmann, 1999, Fiore et al., 1999, Staton, 2007, Harper et al., 1993].

First observe that exponentiation of a presheaf $\mathcal{F} : \widehat{\text{Ren}}_\Sigma$ by a representable has a simpler characterization using the Yoneda lemma (in fact, this works for any base category that has finite products):

$$\begin{aligned} \mathcal{F}^{\mathbf{y}\Delta}(\Gamma) &\cong \widehat{\text{Ren}}_\Sigma[\mathbf{y}\Gamma \times \mathbf{y}\Delta, \mathcal{F}] \\ &\cong \widehat{\text{Ren}}_\Sigma[\mathbf{y}(\Gamma \times \Delta), \mathcal{F}] \\ &\cong \mathcal{F}(\Gamma \times \Delta) \end{aligned}$$

Writing $\mathcal{V}(\tau)$ for the representable presheaf $\mathbf{y}\tau : \widehat{\text{Ren}}_\Sigma$ of variables, we can equivalently use a higher-order notation for terms from inside the topos, with constructors like the following:

$$\begin{aligned} \mathbf{v} &: \mathcal{V}(\tau) \rightarrow \mathfrak{Ne}_\tau \\ \lambda^\sigma &: (\mathcal{V}(\sigma) \rightarrow \mathfrak{Nf}_\tau) \rightarrow \mathfrak{Nf}_{\sigma \rightarrow \tau} \\ &\dots \end{aligned}$$

This is justified by the fact that all the generators of \mathfrak{Sm} , \mathfrak{Ne} and \mathfrak{Nf} commute with the presheaf renaming action. When working internally, we will implicitly use these notations as a simplifying measure.

We will also employ the internal substitution constructors $[\] : \mathbf{1} \rightarrow \mathfrak{Nf}_{[\]}$ and $[\] : \mathbf{1} \rightarrow \mathfrak{Ne}_{[\]}$ defined as follows:

$$\begin{aligned} [\]_\Gamma(\star) &= \mathbf{p}^{[\]} \\ [\]_\Gamma(\star) &= \mathbf{p}^{[\]} \end{aligned}$$

3.3 The gluing construction

Next, we will construct the *gluing category* which will serve as our principal semantic domain for the model construction. Consider the comma category $\text{Gl}_\Sigma \equiv \widehat{\text{Ren}}_\Sigma \downarrow \mathfrak{Sm}$, which “glues” syntactic contexts together with their semantics in presheaves.¹ Con-

¹Careful readers will note that this is a notation for the actual instance of the comma construction, $\text{id}_{\widehat{\text{Ren}}_\Sigma} \downarrow \mathfrak{Sm}$.

cretely, an object of Gl_Σ is a tuple $(\mathcal{D} : \widehat{\text{Ren}}_\Sigma, \Delta : \text{Cl}_\Sigma, \text{quo}_\Delta : \mathcal{D} \rightarrow \mathfrak{M}(\Delta))$; a morphism $(\mathcal{G}, \Gamma, \text{quo}_\Gamma) \rightarrow (\mathcal{D}, \Delta, \text{quo}_\Delta)$ is a commuting square of the following form, which we will suggestively write $\delta \Vdash d$:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{d} & \mathcal{D} \\ \text{quo}_\Gamma \downarrow & & \downarrow \text{quo}_\Delta \\ \mathfrak{M}(\Gamma) & \xrightarrow{\delta} & \mathfrak{M}(\Delta) \end{array}$$

The gluing category Gl_Σ is the category of proof-relevant logical predicates, and is known to be cartesian closed, and thence a model of simply typed lambda calculus. To use this information to our advantage, we will need to “unearth” its cartesian closed structure in explicit terms.

Presentation as a pullback Following Frey [2013], we can give a more intuitive presentation of the gluing construction as a pullback of the fundamental fibration along \mathfrak{M} :

$$\begin{array}{ccc} \text{Gl}_\Sigma & \overset{\text{---}}{\dashrightarrow} & \widehat{\text{Ren}}_\Sigma \\ \text{gl}_\Sigma \downarrow & \lrcorner & \downarrow \text{cod} \\ \text{Cl}_\Sigma & \xrightarrow{\mathfrak{M}} & \widehat{\text{Ren}}_\Sigma \end{array}$$

From the pullback above, we have the *gluing fibration* $\text{gl}_\Sigma : \text{Gl}_\Sigma \rightarrow \text{Cl}_\Sigma$ which acts on objects $(\mathcal{D}, \Delta, \text{quo}_\Delta)$ by projecting Δ , and on morphisms $\delta \Vdash d : \text{Gl}_\Sigma[(\mathcal{G}, \Gamma, \text{quo}_\Gamma), (\mathcal{D}, \Delta, \text{quo}_\Delta)]$ by projecting δ .²

3.4 Reification, reflection and logical predicates

Observe that there are obvious natural embeddings $\text{Rnf}_\tau : \mathfrak{Nf}_\tau \hookrightarrow \mathfrak{M}(\tau)$ and $\text{Rne}_\tau : \mathfrak{Ne}_\tau \hookrightarrow \mathfrak{M}(\tau)$ for each $\tau \in \widehat{\mathcal{U}}_\Sigma$, called “readback”.

In order to give an explicit character to the cartesian closed structure of Gl_Σ , we will define a proof-relevant family of logical predicates $\mathcal{R}_\tau : \widehat{\text{Ren}}_\Sigma$ by induction on $\tau \in \widehat{\mathcal{U}}_\Sigma$, simultaneously exhibiting natural transformations $\uparrow^\tau : \mathfrak{Ne}_\tau \rightarrow \mathcal{R}_\tau$ (pronounced “reflect”) and $\downarrow^\tau : \mathcal{R}_\tau \rightarrow \mathfrak{Nf}_\tau$ (pronounced “reify”) such that the following triangle commutes:

$$\begin{array}{ccc} \mathfrak{Ne}_\tau & \xrightarrow{\downarrow^\tau \circ \uparrow^\tau} & \mathfrak{Nf}_\tau \\ \text{Rne}_\tau \searrow & & \swarrow \text{Rnf}_\tau \\ & \mathfrak{M}(\tau) & \end{array} \quad \text{(reify-reflect yoga)}$$

²Streicher [1998] calls this the “codomain functor”, but to avoid confusion with the codomain functor that it is a pullback of, we use a different terminology.

Remark. An alternative to this approach is to follow Altenkirch et al. [1995] and employ an ad hoc “twisted gluing” category, in which the data of the gluing objects contains the reification and reflection maps. This has the benefit of leading to a proof which is more self-contained, but the disadvantage is that it is not clear how to connect this twisted gluing category to the classical construction.

Atomic types For an atomic type $\sigma \in \mathcal{U}_\Sigma$, we define $\mathcal{R}_\sigma = \mathfrak{Nf}_\sigma$, $\uparrow^\tau = 1$, $\downarrow^\tau = 1$; it is easy to see that the reify-reflect yoga is upheld. Next, we come to the compound types.

Product types Fixing types $\sigma, \tau \in \widehat{\mathcal{U}}_\Sigma$, we define the logical predicate and the reflection and reification maps, using the internal language of $\widehat{\text{Ren}}_\Sigma$:

$$\begin{aligned}\mathcal{R}_{\sigma \times \tau} &= \mathcal{R}_\sigma \times \mathcal{R}_\tau \\ \uparrow^{\sigma \times \tau}(t) &= (\uparrow^\sigma(t.1), \uparrow^\tau(t.2)) \\ \downarrow^{\sigma \times \tau}(v_0, v_1) &= (\downarrow^\sigma(v_0), \downarrow^\tau(v_1))\end{aligned}$$

To execute the reify-reflect yoga, working internally, we fix $t : \mathfrak{Nf}_{\sigma \times \tau}$; we need to observe that $\text{Rnf}_{\sigma \times \tau}(\uparrow^{\sigma \times \tau}(\downarrow^{\sigma \times \tau}(t))) = \text{Rne}_{\sigma \times \tau}(t)$.

$$\begin{aligned}\text{Rnf}_{\sigma \times \tau}(\downarrow^{\sigma \times \tau}(\uparrow^{\sigma \times \tau}(t))) &= \text{Rnf}_{\sigma \times \tau}(\downarrow^{\sigma \times \tau}(\uparrow^\sigma(t.1), \uparrow^\tau(t.2))) \\ &= \text{Rnf}_{\sigma \times \tau}(\downarrow^\sigma(\uparrow^\sigma(t.1)), \downarrow^\tau(\uparrow^\tau(t.2))) \\ &= (\text{Rnf}_\sigma(\downarrow^\sigma(\uparrow^\sigma(t.1))), \text{Rnf}_\tau(\downarrow^\tau(\uparrow^\tau(t.1)))) \\ &= (\text{Rne}_\sigma(t.1), \text{Rne}_\tau(t.2)) && \text{(i.h., i.h.)} \\ &= ((\text{Rne}_{\sigma \times \tau}(t)).1, (\text{Rne}_{\sigma \times \tau}(t)).2) \\ &= \text{Rne}_{\sigma \times \tau}(t) && \text{(PAIR/ETA)}\end{aligned}$$

Above, the steps that commute readback of (neutrals, normals) with the syntax of the λ -theory follow from the fact that normals and neutrals actually embed directly into the syntax unchanged.

Function types To interpret function types, we cannot simply use the exponential in $\widehat{\text{Ren}}_\Sigma$, as this would take us outside the realm of *definable* functions. In a move apparently inspired by Kreisel’s *modified realizability*, we include in the logical predicate both a definable function and its meaning, taking the pullback

$$\begin{array}{ccc}\mathcal{R}_{\sigma \rightarrow \tau} & \dashrightarrow & \mathcal{R}_\tau^{\mathcal{R}_\sigma} \\ \downarrow & \lrcorner & \downarrow \phi \\ \mathfrak{M}(\sigma \rightarrow \tau) & \xrightarrow{\psi} & (\mathfrak{M}(\tau))^{\mathcal{R}_\sigma}\end{array}$$

where for clarity, we define arrows ϕ, ψ in the internal language of $\widehat{\text{Ren}}_\Sigma$ as follows:

$$\begin{aligned}F : \mathcal{R}_\tau^{\mathcal{R}_\sigma} \vdash \phi &\equiv \lambda v. \text{Rnf}_\tau(\downarrow^\tau(F(v))) \\ t : \mathfrak{M}(\sigma \rightarrow \tau) \vdash \psi &\equiv \lambda v. t(\text{Rnf}_\sigma(\downarrow^\sigma(v)))\end{aligned}$$

Abusing notation slightly, we will write an element of $\mathcal{R}_{\sigma \rightarrow \tau}$ as $t \Vdash F$ where $t : \mathfrak{Sm}(\sigma \rightarrow \tau)$ and $F : \mathcal{R}_\tau^{\mathcal{R}_\sigma}$. Next, we need to define reflection of neutrals and reification into normals:

$$\begin{aligned}\uparrow^{\sigma \rightarrow \tau}(t) &= \text{Rne}_{\sigma \rightarrow \tau}(t) \Vdash \lambda v. \uparrow^\tau(t(\downarrow^\sigma(v))) \\ \downarrow^{\sigma \rightarrow \tau}(t \Vdash F) &= \lambda^\sigma(\lambda x. \downarrow^\tau(F(\uparrow^\sigma(v(x))))))\end{aligned}$$

To prove the reify-reflect yoga, (working internally) fix $t : \mathfrak{Ne}_{\sigma \rightarrow \tau}$.

$$\begin{aligned}\text{Rnf}_{\sigma \rightarrow \tau}(\downarrow^{\sigma \rightarrow \tau}(\uparrow^{\sigma \rightarrow \tau}(t))) &= \text{Rnf}_{\sigma \rightarrow \tau}(\downarrow^{\sigma \rightarrow \tau}(\text{Rne}_{\sigma \rightarrow \tau}(t) \Vdash \lambda v. \uparrow^\tau(t(\downarrow^\sigma(v)))))) \\ &= \text{Rnf}_{\sigma \rightarrow \tau}(\lambda^\sigma(\lambda x. \downarrow^\tau(\uparrow^\tau(t(\downarrow^\sigma(\uparrow^\sigma(v(x)))))))) \\ &= \lambda^\sigma(\lambda x. \text{Rnf}_\tau(\downarrow^\tau(\uparrow^\tau(t(\downarrow^\sigma(\uparrow^\sigma(v(x)))))))) \\ &= \lambda^\sigma(\lambda x. \text{Rne}_\tau(t(\downarrow^\sigma(\uparrow^\sigma(v(x)))))) \quad (\text{i.h.}) \\ &= \lambda^\sigma(\lambda x. (\text{Rne}_\tau(t))(\text{Rnf}_\sigma(\downarrow^\sigma(\uparrow^\sigma(v(x)))))) \\ &= \lambda^\sigma(\lambda x. (\text{Rne}_\tau(t))(\text{Rne}_\sigma(v(x)))) \\ &= \lambda^\sigma(\lambda x. (\text{Rne}_\tau(t))(v(x))) \quad (\text{i.h.}) \\ &= \text{Rne}_\tau(t) \quad (\text{ABS/ETA})\end{aligned}$$

Contexts The interpretation is now extended to contexts $\Gamma \in \widetilde{\mathcal{U}}_\Sigma^*$, which are the “types” of substitutions; the interpretation is essentially the same as the one for products.

$$\begin{aligned}\mathcal{R}_{[\]} &= \mathbf{1} \\ \uparrow^{[\]}([\]) &= \star \\ \downarrow^{[\]}(\star) &= [\] \\ \mathcal{R}_{\Gamma, \tau} &= \mathcal{R}_\Gamma \times \mathcal{R}_\tau \\ \uparrow^{\Gamma, \tau}(\gamma.t) &= (\uparrow^\Gamma(\gamma), \uparrow^\tau(t)) \\ \downarrow^{\Gamma, \tau}(g, v) &= \downarrow^\Gamma(g). \downarrow^\tau(v)\end{aligned}$$

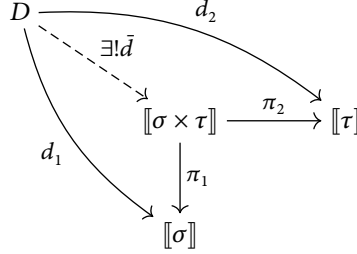
The reify-reflect yoga follows in exactly the same way as it did for products.

Observe that for any $\tau \in \widetilde{\mathcal{U}}_\Sigma$, the triple $\llbracket \tau \rrbracket \equiv (\mathcal{R}_\tau, \tau, \text{quo}_\tau \equiv \text{Rnf}_\tau \circ \downarrow^\tau)$ is an object in Gl_Σ . This brings us to an explicit characterization of the cartesian closed structure of Gl_Σ .

Theorem 3.1. For $\sigma, \tau \in \widetilde{\mathcal{U}}_\Sigma$, $\llbracket \sigma \times \tau \rrbracket$ is the cartesian product $\llbracket \sigma \rrbracket \times \llbracket \tau \rrbracket$ in Gl_Σ .

Proof. We will establish that $\llbracket \sigma \times \tau \rrbracket$ is the cartesian product $\llbracket \sigma \rrbracket \times \llbracket \tau \rrbracket$ by exhibiting its universal property. We need to exhibit a span in Gl_Σ with the following property for any

D, d_1, d_2 :



The projections π_1, π_2 are the following commuting squares:

$$\begin{array}{ccc}
 \mathcal{R}_\sigma \times \mathcal{R}_\tau & \xrightarrow{(p, q) \vdash p} & \mathcal{R}_\sigma \\
 \text{quo}_{\sigma \times \tau} \downarrow & & \text{quo}_\sigma \downarrow \\
 \mathfrak{Im}(\sigma \times \tau) & \xrightarrow{t \vdash t.1} & \mathfrak{Im}(\sigma)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{R}_\sigma \times \mathcal{R}_\tau & \xrightarrow{(p, q) \vdash q} & \mathcal{R}_\tau \\
 \text{quo}_{\sigma \times \tau} \downarrow & & \text{quo}_\tau \downarrow \\
 \mathfrak{Im}(\sigma \times \tau) & \xrightarrow{t \vdash t.2} & \mathfrak{Im}(\tau)
 \end{array}$$

We show that the first square commutes (the second is identical); fixing $p : \mathcal{R}_\sigma, q : \mathcal{R}_\tau$, we calculate.

$$\begin{aligned}
 (\text{Rnf}_{\sigma \times \tau}(\downarrow^{\sigma \times \tau}(p, q))).1 &= (\text{Rnf}_{\sigma \times \tau}(\downarrow^\sigma(p), \downarrow^\tau(q))).1 && \text{(def.)} \\
 &= (\text{Rnf}_\sigma(\downarrow^\sigma(p)), \text{Rnf}_\tau(\downarrow^\tau(q))).1 && \text{(def.)} \\
 &= (\text{Rnf}_\sigma(\downarrow^\sigma(p))) && \text{(FST/BETA)}
 \end{aligned}$$

Next, we need to show that there is a unique mediating arrow $\bar{d} : D \rightarrow \llbracket \sigma \times \tau \rrbracket$ such that the two triangles commute. Unfolding what we are given, we have $D \equiv (\mathcal{D}, \Delta, \text{quo}_\Delta)$ and two commuting squares:

$$\begin{array}{ccc}
 \mathcal{D} & \xrightarrow{d_{00}} & \mathcal{R}_\sigma \\
 \text{quo}_\Delta \downarrow & & \text{quo}_\sigma \downarrow \\
 \mathfrak{Im}(\Delta) & \xrightarrow{d_{01}} & \mathfrak{Im}(\sigma)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{D} & \xrightarrow{d_{10}} & \mathcal{R}_\tau \\
 \text{quo}_\Delta \downarrow & & \text{quo}_\tau \downarrow \\
 \mathfrak{Im}(\Delta) & \xrightarrow{d_{11}} & \mathfrak{Im}(\tau)
 \end{array}$$

We define the mediating map \bar{d} as the following square:

$$\begin{array}{ccc}
 \mathcal{D} & \xrightarrow{(d_{00}, d_{10})} & \mathcal{R}_{\sigma \times \tau} \\
 \text{quo}_\Delta \downarrow & & \text{quo}_{\sigma \times \tau} \downarrow \\
 \mathfrak{Im}(\Delta) & \xrightarrow{(d_{01}, d_{11})} & \mathfrak{Im}(\sigma \times \tau)
 \end{array}$$

To see that the square commutes, fix $p : \mathcal{D}$ and calculate.

$$\begin{aligned}
 \text{Rnf}_{\sigma \times \tau}(\downarrow^{\sigma \times \tau}(d_{00}(p), d_{10}(p))) &= \text{Rnf}_{\sigma \times \tau}(\downarrow^\sigma(d_{00}(p)), \downarrow^\tau(d_{10}(p))) \\
 &= (\text{Rnf}_\sigma(\downarrow^\sigma(d_{00}(p))), \text{Rnf}_\tau(\downarrow^\tau(d_{10}(p)))) \\
 &= (d_{01}(\text{quo}_\Delta(p)), d_{11}(\text{quo}_\Delta(p)))
 \end{aligned}$$

It is easy to see that $\pi_1 \circ \bar{d} = d_1$ and $\pi_2 \circ \bar{d} = d_2$. The uniqueness of \bar{d} with this property follows from the fact that its components are unique: (d_{00}, d_{10}) is the unique mediating arrow given by the universal property of the product $\mathcal{R}_{\sigma \times \tau} \equiv \mathcal{R}_\sigma \times \mathcal{R}_\tau$; moreover, because \mathfrak{M} preserves finite products, we can say the same of (d_{01}, d_{11}) .³ \square

Exercise 3.2. For $\sigma, \tau \in \overline{\mathcal{U}}_\Sigma$, show that $\llbracket \sigma \rightarrow \tau \rrbracket$ is the exponential in $\llbracket \tau \rrbracket^{\llbracket \sigma \rrbracket}$ in Gl_Σ [Johnstone, 2002, Example 2.1.12].

Corollary 3.3. Gl_Σ is a model of the free λ -theory generated by Σ , with interpretation functor $\llbracket - \rrbracket : \text{Cl}_\Sigma \rightarrow \text{Gl}_\Sigma$.

Theorem 3.4. The composite functor $\text{gl}_\Sigma \circ \llbracket - \rrbracket$ is the identity endofunctor on Cl_Σ :

$$\begin{array}{ccc} \text{Cl}_\Sigma & \xrightarrow{\llbracket - \rrbracket} & \text{Gl}_\Sigma \\ & \searrow \text{id}_{\text{Cl}_\Sigma} & \downarrow \text{gl}_\Sigma \\ & & \text{Cl}_\Sigma \end{array}$$

Proof. This follows immediately from the fact that Cl_Σ is the classifying category of the theory Σ , so it is the initial category with the structure of Σ . Therefore, any Σ -homomorphism $\text{Cl}_\Sigma \rightarrow \text{Cl}_\Sigma$ must be the identity, including the composite above. \square

Now, working externally in the category \mathfrak{Set} , we can explicitly construct the normalization function, $\text{nf}_\Gamma^\Delta : \text{Cl}_\Sigma[\Gamma, \Delta] \rightarrow \mathfrak{Nf}_\Delta(\Gamma)$ as the following composite:

$$\begin{array}{ccccccc} & & & \text{nf}_\Gamma^\Delta & & & \\ & & & \text{-----} & & & \\ \text{Cl}_\Sigma[\Gamma, \Delta] & \xrightarrow{\llbracket - \rrbracket} & \text{Gl}_\Sigma[\llbracket \Gamma \rrbracket, \llbracket \Delta \rrbracket] & \xrightarrow{\pi} & \widehat{\text{Ren}}_\Sigma[\mathcal{R}_\Gamma, \mathcal{R}_\Delta] & \xrightarrow{\phi} & \mathcal{R}_\Delta(\Gamma) & \xrightarrow{\downarrow \Delta} & \mathfrak{Nf}_\Delta(\Gamma) \end{array}$$

where

$$\phi(v) = v_\Gamma(\uparrow_\Gamma^\Gamma(\text{id}_\Gamma^\Gamma))$$

Theorem 3.5 (Completeness). If $\Gamma \vdash_\Sigma t_0 = t_1 : \tau$, then $\Gamma \vdash_\Sigma \text{Rnf}_\tau^\Gamma(\text{nf}_\Gamma^\tau(t_0)) = \text{Rnf}_\tau^\Gamma(\text{nf}_\Gamma^\tau(t_1)) : \tau$.

Proof. This is immediate from the fact that we have defined a function out of the morphisms of Cl_Σ , which are already quotiented under definitional equivalence. \square

Theorem 3.6 (Normalization). If $\Gamma \vdash_\Sigma t : \tau$, then $\Gamma \vdash_\Sigma \text{Rnf}_\tau^\Gamma(\text{nf}_\Gamma^\tau(t)) = t : \tau$.

³Recall that \mathfrak{M} is defined as $i^* \circ y$ with i^* the reindexing functor induced by $i : \text{Ren}_\Sigma \rightarrow \text{Cl}_\Sigma$. Because i^* has a left adjoint, given by Kan extension, it preserves limits; because the Yoneda embedding also preserves limits, \mathfrak{M} preserves limits too.

Proof. Suppose $\llbracket t \rrbracket = (t_0 \Vdash v)$. Now calculate.

$$\begin{aligned}
\text{Rnf}_\tau^\Gamma(\mathbf{nf}_\Gamma^\tau(t)) &= \text{Rnf}_\tau^\Gamma(\downarrow_\Gamma^\tau(v_\Gamma(\uparrow_\Gamma^\Gamma(\text{id}_\Gamma^\Gamma)))) \\
&= (\text{Rnf}_\tau \circ \downarrow^\tau \circ v)_\Gamma(\uparrow_\Gamma^\Gamma(\text{id}_\Gamma^\Gamma)) \\
&= (t_0 \circ \text{Rnf}_\Gamma \circ \downarrow^\Gamma)_\Gamma(\uparrow_\Gamma^\Gamma(\text{id}_\Gamma^\Gamma)) && \text{(comma condition)} \\
&= t_0^\Gamma(\text{Rnf}_\Gamma^\Gamma(\downarrow_\Gamma^\Gamma(\uparrow_\Gamma^\Gamma(\text{id}_\Gamma^\Gamma)))) \\
&= t_0^\Gamma(\text{Rne}_\Gamma^\Gamma(\text{id}_\Gamma^\Gamma)) && \text{(reify-reflect yoga)} \\
&= t_0^\Gamma(\text{id}_\Gamma^\Gamma)
\end{aligned}$$

$\Gamma \vdash_\Sigma t_0^\Gamma(\text{id}_\Gamma^\Gamma) = t : \tau$. Writing $\llbracket t \rrbracket$ for the induced natural transformation $\mathfrak{Sm}(\Gamma) \rightarrow \mathfrak{Sm}(\tau)$ such that $\Gamma \vdash_\Sigma \llbracket t \rrbracket_\Gamma(\text{id}_\Gamma^\Gamma) = t : \tau$, it suffices to show that $t = t_0$. Because $\text{gl}_\Sigma(\llbracket t \rrbracket) = t_0$, by Theorem 3.4 we have $t = t_0$. \square

Corollary 3.7 (Soundness). *If $\Gamma \vdash_\Sigma \text{Rnf}_\tau^\Gamma(\mathbf{nf}_\Gamma^\tau(t_0)) = \text{Rnf}_\tau^\Gamma(\mathbf{nf}_\Gamma^\tau(t_1)) : \tau$, then $\Gamma \vdash_\Sigma t_0 = t_1 : \tau$.*

Proof. To see that $t_0 = t_1$, observe that by Theorem 3.6 we have $\text{Rnf}_\tau^\Gamma(\mathbf{nf}_\Gamma^\tau(t_i)) = t_i$, so by transitivity and assumption we have $t_0 = t_1$. \square

4 Perspective

4.1 Global sections and the Freyd cover

A more common use of the gluing technique lies in the construction of the Freyd cover (also called the “scone”, which is short for “Sierpinski cone”) of a topos in order to prove properties of *closed proofs* in intuitionistic higher-order logic, such as the disjunction and existence properties, which correspond in λ -calculus to instances of the closed canonicity result.

As an example, we will prove both these properties for intuitionistic higher-order logic over simple types and the natural numbers.⁴ Writing \mathcal{F} for the free topos generated by a natural numbers object N , observe that we have the global sections functor $\Gamma \equiv \mathcal{F}(1, -)$ which takes every object to its global elements. We define the Freyd cover $\check{\mathcal{F}}$ over \mathcal{F} as the gluing category obtained by pulling back the fundamental fibration along the global sections functor:

$$\begin{array}{ccc}
\check{\mathcal{F}} & \dashrightarrow & \mathfrak{Set}^{\rightarrow} \\
\pi_1 \downarrow & & \downarrow \text{cod} \\
\mathcal{F} & \xrightarrow{\Gamma} & \mathfrak{Set}
\end{array}$$

⁴This section is an expanded version of material which appears in Shulman [2006], with some more details filled in.

Because the global sections functor preserves finite limits, the Freyd cover $\check{\mathcal{F}}$ is again a topos with π_1 a *logical functor* [Johnstone, 2002, Example 2.1.12]; moreover, π_1 preserves the natural numbers object [Taylor, 1999, Corollary 7.7.2]. We also have a functor $\pi_0 : \check{\mathcal{F}} \rightarrow \mathbf{Set}$, which merely preserves finite limits.

The Freyd cover $\check{\mathcal{F}}$ has a natural numbers object $\check{\mathbb{N}}$ given by $(\mathbb{N}, \mathbb{N}, n \mapsto \bar{n})$, where \bar{n} takes a set-theoretic natural number to the corresponding global section in \mathcal{F} , and π_1 preserves the natural numbers object. Because \mathcal{F} is the initial topos with a natural numbers object, for any other such topos \mathcal{E} we have a *unique* map $I_{\mathcal{E}} : \mathcal{F} \rightarrow \mathcal{E}$.

Lemma 4.1. *The logical functor $\pi_1 : \check{\mathcal{F}} \rightarrow \mathcal{F}$ is a retract of $I_{\check{\mathcal{F}}}$:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{I_{\check{\mathcal{F}}}} & \check{\mathcal{F}} \\ & \searrow 1 & \downarrow \pi_1 \\ & & \mathcal{F} \end{array}$$

Proof. We have the “additional” identity morphism $1 : \mathcal{F} \rightarrow \mathcal{F}$, so by initiality of \mathcal{F} , we must have $1 = \pi_1 \circ I_{\check{\mathcal{F}}}$. \square

Theorem 4.2 (Natural number canonicity). *Any global section $n : \Gamma(\mathbb{N})$ in \mathcal{F} is equal to some numeral \bar{k} .*

Proof. The functor $I_{\check{\mathcal{F}}}$ necessarily preserves \mathbb{N} . Therefore, the global section n lifts in $\check{\mathcal{F}}$ to a square in \mathbf{Set} as follows:

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow \\ \Gamma(\mathbf{1}) \cong \mathbf{1} & \longrightarrow & \Gamma(\mathbb{N}) \end{array}$$

The upstairs morphism gives us a numeral k ; because the diagram commutes and using Lemma 4.1, we have $n = \bar{k}$. \square

Theorem 4.3 (Existence property). *Suppose that $\mathcal{F} \vDash \exists x : X.\phi(x)$; then there is a global element $\alpha : \mathbf{1} \rightarrow X$ in \mathcal{F} such that $\mathcal{F} \vDash \phi(\alpha)$.*

Proof. We will use the Kripke-Joyal semantics of the topos [Mac Lane and Moerdijk, 1992]; unwinding our assumption $\mathbf{1} \Vdash \exists x : X.\phi(x)$, we have that there exists an epimorphism $p : V \twoheadrightarrow \mathbf{1}$ and a morphism $\beta : V \rightarrow X$ such that $V \Vdash \phi(\beta)$.⁵

The logical functor $I_{\check{\mathcal{F}}} : \mathcal{F} \rightarrow \check{\mathcal{F}}$ lifts p to an epimorphism $I_{\check{\mathcal{F}}}(p) : I_{\check{\mathcal{F}}}(V) \twoheadrightarrow I_{\check{\mathcal{F}}}(\mathbf{1})$ in $\check{\mathcal{F}}$. Since $I_{\check{\mathcal{F}}}$ preserves the terminal object, this is actually to say $I_{\check{\mathcal{F}}}(p) : I_{\check{\mathcal{F}}}(V) \twoheadrightarrow \mathbf{1}$ in $\check{\mathcal{F}}$. $I_{\check{\mathcal{F}}}(p)$ must be a square in \mathbf{Set} of the following kind:

$$\begin{array}{ccc} \pi_0(I_{\check{\mathcal{F}}}(V)) & \twoheadrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \Gamma(V) & \longrightarrow & \Gamma(\mathbf{1}) \end{array}$$

⁵Please note that the symbol \Vdash here denotes the forcing relation, rather than the gluing relation.

Because the upstairs morphism is a surjection, we know that $\pi_0(I_{\check{\mathcal{F}}}(V))$ is non-empty; therefore, because we have a map $\pi_0(I_{\check{\mathcal{F}}}(V)) \rightarrow \Gamma(V)$, we can see that $\Gamma(V)$ is non-empty, i.e. we have a global section of $\rho : \mathbf{1} \rightarrow V$ in \mathcal{F} . By precomposition and Kripke-Joyal monotonicity, then, we have a global section $\beta \circ \rho : \mathbf{1} \rightarrow X$ such that $\mathbf{1} \Vdash \phi(\beta \circ \rho)$. \square

Theorem 4.4 (Disjunction property). *Suppose that $\mathcal{F} \vDash \phi(\alpha) \vee \psi(\alpha)$ for some $\alpha : \mathbf{1} \rightarrow X$; then either $\mathcal{F} \vDash \phi(\alpha)$ or $\mathcal{F} \vDash \psi(\alpha)$.*

Proof. We will use essentially the same technique as in our proof of Theorem 4.3. Unwinding the Kripke-Joyal semantics of the topos, we have morphisms $p : V \rightarrow \mathbf{1}$ and $q : W \rightarrow \mathbf{1}$ such that $[p, q] : V + W \rightarrow \mathbf{1}$ is an epimorphism and moreover $V \Vdash \phi(\alpha \circ p)$ and $W \Vdash \psi(\alpha \circ q)$. As above, $[p, q]$ lifts to an epimorphism in $\check{\mathcal{F}}$ as follows:

$$\begin{array}{ccc} \pi_0(I_{\check{\mathcal{F}}}(V + W)) = \pi_0(I_{\check{\mathcal{F}}}(V)) + \pi_0(I_{\check{\mathcal{F}}}(W)) & \twoheadrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \Gamma(V + W) = \Gamma(V) + \Gamma(W) & \longrightarrow & \mathbf{1} \end{array}$$

Note that the global sections functor for \mathcal{F} preserves finite colimits, and π_0 preserves all colimits [Taylor, 1999, Proposition 7.7.1(l)]. Because the upstairs morphism is a surjection, we know that $\pi_0(I_{\check{\mathcal{F}}}(V)) + \pi_0(I_{\check{\mathcal{F}}}(W))$ is non-empty, whence we must have either a global section $r \in \Gamma(V)$ or a global section $s \in \Gamma(W)$.

Supposing we have a global section $r : \mathbf{1} \rightarrow V$ in \mathcal{F} , then by Kripke monotonicity, we have $\mathbf{1} \Vdash \phi(\alpha \circ p \circ r)$. On the other hand, if we have a global section $s : \mathbf{1} \rightarrow W$, then we would have $\mathbf{1} \Vdash \psi(\alpha \circ q \circ s)$. \square

4.2 Connection with the method of computability

As we have alluded to in the previous section, the gluing category always functions as the “category of suitable logical predicates”, with the meaning of “suitable” negotiated by choice of gluing functor. Most instances of the logical relations/predicates technique can be phrased as an instance of the more general gluing construction.

4.2.1 Proof (ir)relevance

The native notion of logical “predicate” which is induced by the gluing construction is a proof-relevant one, whereas in the method of computability, one generally studies predicates in the classical, proof-irrelevant sense. This restriction is easily accounted for by making a slight adjustment to the categories involved.

Writing \mathbb{C} for the classifying category of our theory, if we take a category \mathcal{E} to be our semantic domain, we can form categories of proof-relevant logical predicates and proof

irrelevant logical predicates respectively along a functor $F : \mathbb{C} \rightarrow \mathcal{E}$ as follows:⁶

$$\begin{array}{ccc}
 \text{Gl} & \xrightarrow{\quad} & \mathcal{E}^{\rightarrow} \\
 \downarrow \lrcorner & & \downarrow \text{cod} \\
 \mathbb{C} & \xrightarrow{F} & \mathcal{E}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{Gl}_{irr} & \xrightarrow{\quad} & \mathcal{E}_{mono}^{\rightarrow} \\
 \downarrow \lrcorner & & \downarrow \mathcal{E}^{\rightarrow} \\
 \mathbb{C} & \xrightarrow{F} & \mathcal{E}
 \end{array}$$

When F is the global sections functor (and thence Gl is the Freyd cover or the scone of \mathbb{C}), Gl_{irr} is often called the “subscone” of \mathbb{C} .

4.2.2 Relations vs predicates

What we have seen so far corresponds to the technique of unary logical relations, but the abstraction scales easily to the case of binary (and n -ary) logical relations by replacing \mathbb{C} with $\mathbb{C} \times \mathbb{D}$, as described in Mitchell and Scedrov [1993]. To see the connection with binary logical relations, it will be instructive to work out explicitly the case for exponentials in the subscone of $\mathbb{C} \times \mathbb{C}$, which we will write $\widetilde{\mathbb{C} \times \mathbb{C}}$.

First, observe that the exponential in the product of two cartesian closed categories is calculated pointwise; so for $(A_0, A_1), (B_0, B_1) : \mathbb{C} \times \mathbb{C}$, we have $(B_0, B_1)^{(A_0, A_1)} = (B_0^{A_0}, B_1^{A_1})$.

An object in $\widetilde{\mathbb{C} \times \mathbb{C}}$ is a monomorphism $R \mapsto \Gamma(A, B)$ where Γ is the global sections functor for $\mathbb{C} \times \mathbb{C}$. Because the global sections functor preserves finite limits, this is to say that we have a monomorphism $R \mapsto \Gamma(A) \times \Gamma(B)$, in other words *a relation on the closed terms of type A and B in the language \mathbb{C}* .

We wish to inspect for ourselves the exponential object in $\widetilde{\mathbb{C} \times \mathbb{C}}$. As we saw earlier on, to form the exponential in the gluing category we first take the following pullback:

$$\begin{array}{ccc}
 E & \xrightarrow{\quad} & S^R \\
 \downarrow \lrcorner & & \downarrow \lrcorner \\
 \Gamma(B_0^{A_0}) \times \Gamma(B_1^{A_1}) & \longrightarrow & (\Gamma(B_0, B_1))^R
 \end{array}$$

Then, we define the exponential $(S \mapsto \Gamma(B_0) \times \Gamma(B_1))^{(R \mapsto \Gamma(A_0) \times \Gamma(A_1))}$ to be the monomorphism on the left. Now, unfolding definitions, a global element of this exponential is simply a pair of closed terms $\cdot \vdash F_0 : A_0 \rightarrow B_0$ and $\cdot \vdash F_1 : A_1 \rightarrow B_1$ together with a function $H : S^R$ which is tracked by (F_0, F_1) ; unwinding further, this means only that for all $\cdot \vdash a_0 : A_0$ and $\cdot \vdash a_1 : A_1$, if $(a_0, a_1) \in R$, then $(F_0(a_0), F_1(a_1)) \in S$.

4.2.3 Kripke/Beth/Grothendieck logical relations

A common generalization of the method of computability is to use a logical relation which is indexed in some partial order (or even a category), subject to a functoriality

⁶For intuition, consider the specific example where \mathcal{E} is \mathfrak{Set} and \mathcal{F} is the global sections functor, as in Section 4.1.

condition. In the literature, these are called *Kripke logical relations*, and indeed, the construction that we used to prove normalization of free λ -theories in Section 3 is the proof-relevant unary Kripke instance of the gluing abstraction, where the worlds are contexts of variables linked by renamings.

Many other variations of indexed logical relations appear in the wild, and nearly all of these are already accounted for within the abstraction. For instance, by imposing Grothendieck topology on the base poset or category and requiring a *local character* condition in addition to monotonicity, one can develop something which might be called *Beth/Grothendieck logical relations* (see Coquand and Manna [2016], Altenkirch et al. [2001] and Fiore and Simpson [1999] for examples).

Remark (Terminology). In the literature [Jung and Tiuryn, 1993, Fiore and Simpson, 1999, Fiore, 2002], the proof irrelevant version of this construction appears under the somewhat confusing name “Kripke Relations of Varying Arity”—confusing because it is not immediately clear what it has to do with the arity of a relation.

In the early literature (such as Jung and Tiuryn [1993]), there was some resistance to explaining what these were in a more conceptual way, but as described in Fiore and Simpson [1999], these have a simple characterization as *internal relations* of a certain kind within a presheaf topos which corresponds exactly to a proof irrelevant version of the construction we describe in these notes.

Example 4.5 (Independence of Markov’s Principle). In Coquand and Manna [2016], the method of computability was used to establish the independence of Markov’s Principle from Martin-Löf Type Theory using a forcing extension over Cantor space \mathcal{C} . We will briefly describe how the construction in that paper fits into the framework of gluing.

Letting \mathbb{C} be the classifying category of the forcing extension of type theory, we have a fibration $\pi_{\mathcal{C}} : \mathbb{C} \rightarrow \mathcal{C}$ which projects the forcing condition (a representation of compact open in Cantor space). Writing $\mathbf{Sh}(\mathcal{C})$ for the topos of sheaves over Cantor space, we evidently have a functor $\mathfrak{M} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbf{Sh}(\mathcal{C})$ which takes a pair of contexts (Δ_0, Δ_1) to the following sheaf:

$$(p : \mathcal{C}) \mapsto \{(\delta_0, \delta_1) \mid p \leq \pi_{\mathcal{C}}(\Delta_0) \wedge p \leq \pi_{\mathcal{C}}(\Delta_1) \wedge \vdash_p \delta_0 : (\Delta_0)_{|p} \wedge \vdash_p \delta_1 : (\Delta_1)_{|p}\}$$

(The above is a sheaf, because the topology on \mathcal{C} is subcanonical, and because the calculus contains a rule for local character.)

Now, consider the gluing category obtained from the following pullback:

$$\begin{array}{ccc} \mathbf{Gl} & \longrightarrow & \mathbf{Sh}(\mathcal{C})_{\text{mono}}^{\rightarrow} \\ \downarrow \lrcorner & & \downarrow \\ \mathbb{C} \times \mathbb{C} & \xrightarrow{F} & \mathbf{Sh}(\mathcal{C}) \end{array}$$

$\downarrow \text{cod}$

Viewed externally, the objects of \mathbf{Gl} are \mathcal{C} -indexed binary relations on closed terms in \mathbb{C} which enjoy both monotonicity and local character. By examining the cartesian closed

structure of GI , it can be seen (as above) that the logical relations for each connective match the naïve ones.

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