Normalization for Cubical Type Theory*

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Cubical type theory [5, 2] is a variant of Homotopy Type Theory [14] that *computes* [8, 3], in the sense that it admits a computable strict isomorphism between the global points of the type $\mathbb{N}$ and the actual natural numbers, a result obstructed for HoTT by its axiomatic treatment of Voevodsky's univalence principle. The computational behavior of cubical type theory is effected by an interval $\mathbb{I}$, maps out of which correspond to paths or identifications between generalized points of types.

Although characterizing the global points of $\mathbb{N}$ is of philosophical and programmatic importance to type theorists and logicians, such *canonicity* results do not have many practical implications for the use of cubical type theory as a basis for computerized proof assistants like Cubical Agda and redtt [15, 10]. For these applications, it is necessary to prove a *normalization* result characterizing the generalized points of all types, which usually moreover implies the decidability of the formal system under consideration.

Going back to Peter Freyd's observation [7] that Artin gluing [4, 16] can be used to establish the existence and disjunction principles for intuitionistic higher order logic, there is a long tradition of using abstract category theoretic methods (including both gluing and realizability) to prove syntactic results about logics and type theories by semantic means [1, 6, 9]. There is moreover a precise relationship between categorical gluing and Tait's method of computability: a logical predicate on a language (in the sense of Tait) can be viewed as a partition of a topos into open and closed subtopoi, where the open subtopos embeds the syntactic category of the language and the closed complement is determined by the desired theorem. For theorems characterizing global points, the closed subtopos is the point; subtler figures arise when characterizing generalized points.

Recently the first author and Robert Harper observed that such arguments can be profitably rephrased *synthetically* in the internal language of a glued topos [13], wherein complementary open and closed modalities are related by an internal fracture theorem [11], and thereby generate all the necessary structure for an abstract computability argument. In the present work, we apply this synthetic version of gluing to establish normalization for univalent Cartesian cubical type theory without universes, which the first author has extended to incorporate a cumulative hierarchy of univalent universes in his doctoral dissertation.

Normalization results for type theory rely on splitting every type A into a "neutral part" $\mathsf{Neutral}(A)$ and a "normal part" $\mathsf{Normal}(A)$, where the former governs the maps out of A and the latter governs the maps into A; for instance, we have $\pi_1 : \mathsf{Neutral}(A \times B) \to \mathsf{Neutral}(A)$ and $\mathsf{pair} : \mathsf{Normal}(A) \times \mathsf{Normal}(B) \to \mathsf{Normal}(A \times B)$. Then, the normal form of an element $b : A \to B$ is simply a morphism $\mathsf{Neutral}(A) \to \mathsf{Normal}(B)$. As a result of this bifurcation, the language of normal forms is completely free: the composite $\pi_1 \circ \mathsf{pair}$ cannot be written, so there is no need for any reduction.

Unfortunately, the presence of the interval in cubical type theory obstructs the classic neutral/normal decomposition because neutrality ceases to be a stable notion: if $p : \mathsf{Neutral}(\mathsf{Path}(a, b))$ is a neutral path from a to b and $i : \mathbb{I}$ is a generic element of the interval, the application $p(i)$ should be neutral but its 0-face $p(0)$, which computes to a, is neither neutral nor normal. Inspired by the behavior of $p(i)$ at $(i = 0) \vee (i = 1)$, we have discovered an essentially geometrical solution to the stability problem: every neutral $e : \mathsf{Neutral}(A)$ determines a *locus of instability* ∂e, along which we glue a compatible partial normal form $e' : \partial e \to \mathsf{Normal}(A)$ to "stabilize" e before composing it into a normal form. The stabilization of neutrals leads to a novel generalization of Tait's famous *saturation* condition, the critical ingredient to extend traditional topos theoretic gluing methods to prove normalization for cubical type theory.

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