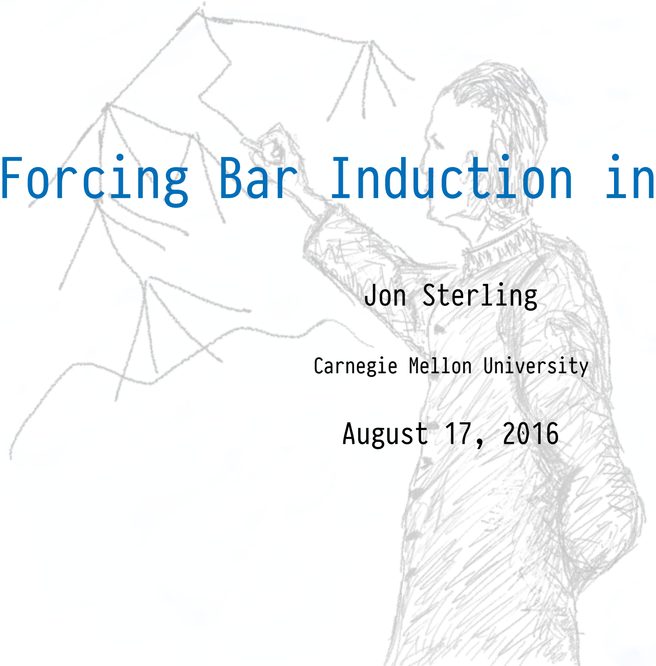


Forcing Bar Induction in **System** $\mathbb{T}$

Jon Sterling

Carnegie Mellon University

August 17, 2016

Brouwer's Bar Thesis

“All bars on a spread can be coded as inductive trees.”

Brouwer's Bar Thesis

"All bars on a spread can be coded as inductive trees."

$\star$ $\Rightarrow$ the Fan Theorem (intuitionistic König's Lemma)

Brouwer's Bar Thesis

"All bars on a spread can be coded as inductive trees."

- $\star$ $\Rightarrow$ the Fan Theorem (intuitionistic König's Lemma)
- $\star$ $\Rightarrow$ all functions on the interval $\mathbb{I} \triangleq [0,1]$ are uniformly continuous

Brouwer's Bar Thesis

"All bars on a spread can be coded as inductive trees."

- ⋆ $\Rightarrow$ the Fan Theorem (intuitionistic König's Lemma)
- ⋆ $\Rightarrow$ all functions on the interval $\mathbb{I} \triangleq [0,1]$ are uniformly continuous
- ⋆ $\Rightarrow$ completeness of intuitionistic first-order logic (Bickford, Constable)

Brouwer's Bar Thesis

"All bars on a spread can be coded as inductive trees."

- $\star$ $\Rightarrow$ the Fan Theorem (intuitionistic König's Lemma)
- $\star$ $\Rightarrow$ all functions on the interval $\mathbb{I} \triangleq [0, 1]$ are uniformly continuous
- $\star$ $\Rightarrow$ completeness of intuitionistic first-order logic (Bickford, Constable)

In general, implies termination for a class of functional programs on infinite trees.

Spreads as spaces and trees

A ***spread*** is at the same time a topological space and a non-wellfounded tree.

Spreads as spaces and trees

A ***spread*** is at the same time a topological space and a non-wellfounded tree.

A spread is defined by a predicate $\mathfrak{S}$ on lists of natural numbers subject to some laws:

Spreads as spaces and trees

A ***spread*** is at the same time a topological space and a non-wellfounded tree.

A spread is defined by a predicate $\mathfrak{S}$ on lists of natural numbers subject to some laws:

1. If $\vec{u} \in \mathfrak{S}$, then there exists an $x \in \mathbb{N}$ such that $\vec{u} \frown x \in \mathfrak{S}$.

Spreads as spaces and trees

A ***spread*** is at the same time a topological space and a non-wellfounded tree.

A spread is defined by a predicate $\mathfrak{S}$ on lists of natural numbers subject to some laws:

1. If $\vec{u} \in \mathfrak{S}$, then there exists an $x \in \mathbb{N}$ such that $\vec{u} \frown x \in \mathfrak{S}$.
2. If $\vec{u} \frown x \in \mathfrak{S}$, then also $\vec{u} \in \mathfrak{S}$.

Spreads as spaces and trees

A *spread* is at the same time a topological space and a non-wellfounded tree.

A spread is defined by a predicate $\mathfrak{S}$ on lists of natural numbers subject to some laws:

1. If $\vec{u} \in \mathfrak{S}$, then there exists an $x \in \mathbb{N}$ such that $\vec{u} \frown x \in \mathfrak{S}$.
2. If $\vec{u} \frown x \in \mathfrak{S}$, then also $\vec{u} \in \mathfrak{S}$.
3. Finally, $\langle\rangle \in \mathfrak{S}$.

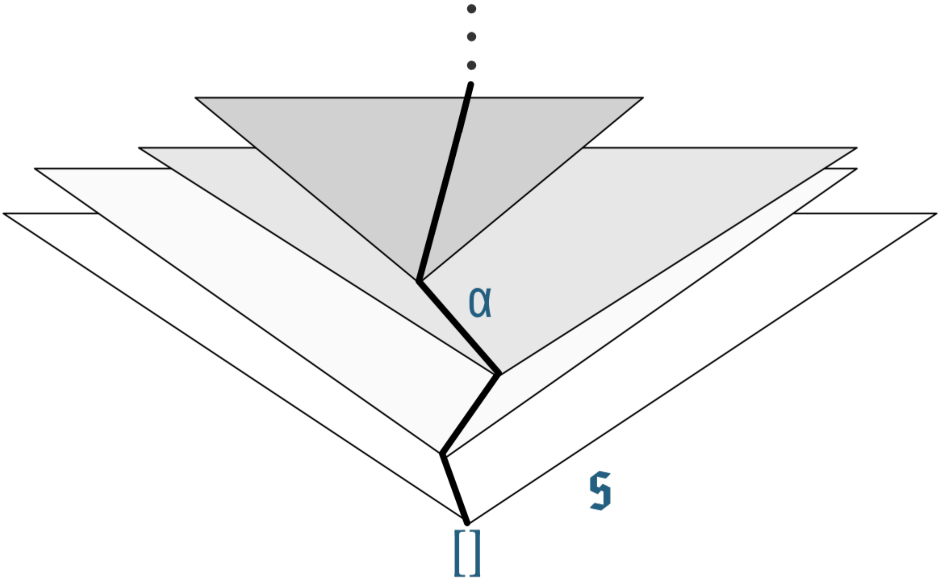
Spreads as spaces and trees

A ***spread*** is at the same time a topological space and a non-wellfounded tree.

A spread is defined by a predicate $\mathfrak{S}$ on lists of natural numbers subject to some laws:

1. If $\vec{u} \in \mathfrak{S}$, then there exists an $x \in \mathbb{N}$ such that $\vec{u}\frown x \in \mathfrak{S}$.
2. If $\vec{u}\frown x \in \mathfrak{S}$, then also $\vec{u} \in \mathfrak{S}$.
3. Finally, $\langle\rangle \in \mathfrak{S}$.

The predicate $\mathfrak{S}$ either defines the finite prefixes of paths down an infinite tree, or it defines the lattice of open sets of a topological space.


α
𝔰
[]

Neighborhoods and Points

A list of naturals $\vec{u} \in \mathbb{N}*$ can be thought of as a neighborhood around a point, or as a prefix of a path through an infinite tree.

A stream of naturals $\alpha \in \mathbb{N}^{\mathbb{N}}$ can be thought of as an ideal point in the spread (space), or as a path through the spread's infinite tree.

$$\vec{u} \prec \alpha \qquad (\vec{u} \text{ approximates } \alpha)$$

$$\alpha \in \vec{u} \qquad (\vec{u} \text{ is a neighborhood around } \alpha)$$

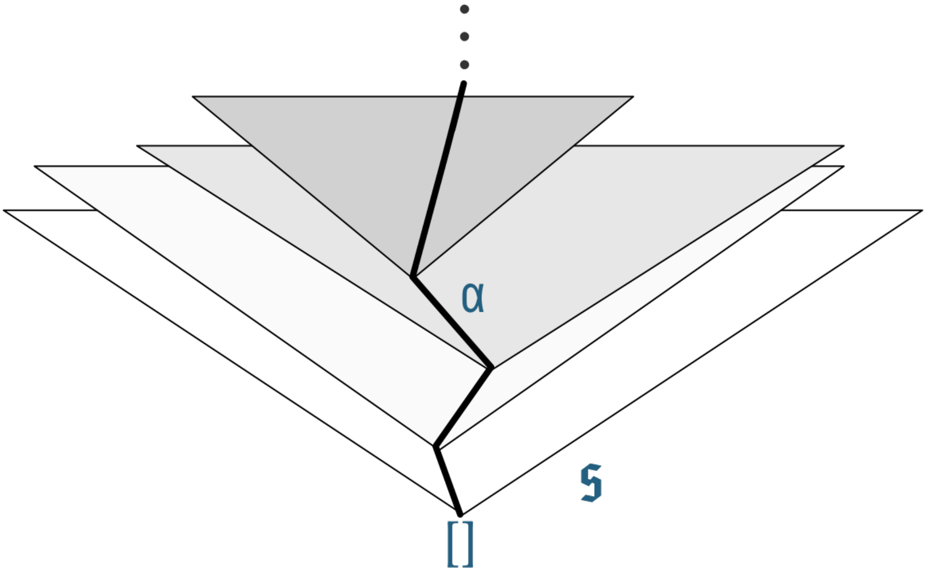
To Bar A Node

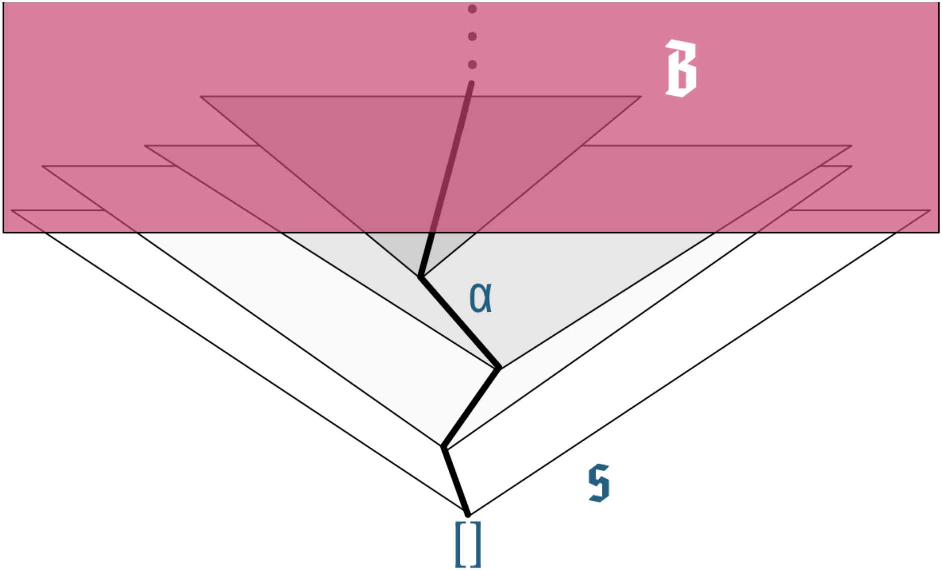
A bar $\mathfrak{B}$ is a predicate on neighborhoods such that every point “hits it”. More generally, $\mathfrak{B}$ bars a neighborhood $\vec{u}$ when every path through $\vec{u}$ ends up in $\mathfrak{B}$.

To Bar A Node

A bar $\mathfrak{B}$ is a predicate on neighborhoods such that every point “hits it”. More generally, $\mathfrak{B}$ bars a neighborhood $\vec{u}$ when every path through $\vec{u}$ ends up in $\mathfrak{B}$.

$$\frac{\forall \alpha \in \vec{u}.\ \exists n \in \mathbb{N}.\ \overline{\alpha}[n] \in \mathfrak{B}}{\vec{u} \lhd \mathfrak{B}}$$


α
𝔰
[]


𝔅
α
𝔰
[]

Inductive Barhood

“All demonstrations of barhood can be analyzed into **inductive mental constructions.**”

Inductive Barhood

"All demonstrations of barhood can be analyzed into **inductive mental constructions.**"

Let $\mathfrak{S}^{\natural}(\vec{u}) \triangleq \left\{ x \in \mathbb{N} \mid \vec{u}^\frown x \in \mathfrak{S} \right\}$. Presupposing $\vec{u} \in \mathfrak{S}$ and $\mathfrak{B}$ *monotone*:

Inductive Barhood

"All demonstrations of barhood can be analyzed into **inductive mental constructions.**"

Let $\mathfrak{S}^\natural(\vec{u}) \triangleq \left\{ x \in \mathbb{N} \mid \vec{u}^\frown x \in \mathfrak{S} \right\}$. Presupposing $\vec{u} \in \mathfrak{S}$ and $\mathfrak{B}$ *monotone*:

$$\frac{\vec{u} \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \eta \qquad \frac{\forall x \in \mathfrak{S}^\natural(\vec{u}).\ \vec{u}^\frown x \blacktriangleleft_{ind} \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \mathsf{F}$$

Inductive Barhood

"All demonstrations of barhood can be analyzed into **inductive mental constructions.**"

Let $\mathfrak{S}^{\natural}(\vec{u}) \triangleq \left\{ x \in \mathbb{N} \mid \vec{u} \frown x \in \mathfrak{S} \right\}$. Presupposing $\vec{u} \in \mathfrak{S}$ and $\mathfrak{B}$ *monotone*:

$$\frac{\vec{u} \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \eta \qquad \frac{\forall x \in \mathfrak{S}^{\natural}(\vec{u}).\ \vec{u} \frown x \blacktriangleleft_{ind} \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \mathsf{F}$$

Admissible (by monotonicity):

$$\frac{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}{\vec{u} \frown x \blacktriangleleft_{ind} \mathfrak{B}}\ \zeta$$

Brouwer's Bar Thesis

Recall $\vec{u} \triangleleft \mathcal{B} \triangleq \forall \alpha \in \vec{u}.\ \exists n \in \mathbb{N}.\ \overline{\alpha}[n] \in \mathcal{B}$.

Brouwer's Bar Thesis

Recall $\vec{u} \lhd \mathfrak{B} \triangleq \forall \alpha \in \vec{u}.\ \exists n \in \mathbb{N}.\ \overline{\alpha}[n] \in \mathfrak{B}$.

Brouwer's Bar Thesis is the ***adequacy*** of the inductive coding of barhood:

$$\frac{\vec{u} \lhd \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \textbf{BT}?$$

Brouwer's Bar Thesis

Recall $\vec{u} \lhd \mathcal{B} \triangleq \forall \alpha \in \vec{u}.\ \exists n \in \mathbb{N}.\ \overline{\alpha}[n] \in \mathcal{B}$.

Brouwer's Bar Thesis is the ***adequacy*** of the inductive coding of barhood:

$$\frac{\vec{u} \blacktriangleleft_{ind} \mathcal{B}}{\vec{u} \lhd \mathcal{B}} \text{ soundness} \qquad \frac{\vec{u} \lhd \mathcal{B}}{\vec{u} \blacktriangleleft_{ind} \mathcal{B}} \text{ completeness?}$$

Brouwer's Bar Thesis

$$\frac{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}{\vec{u} \triangleleft \mathfrak{B}} \text{ soundness} \qquad \frac{\vec{u} \triangleleft \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}} \text{ completeness?}$$

Brouwer's Bar Thesis

$$\frac{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}{\vec{u} \triangleleft \mathfrak{B}} \text{ soundness} \qquad \frac{\vec{u} \triangleleft \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}} \text{ completeness?}$$

⋆ soundness is easy! Just count up the F-nodes.

Brouwer's Bar Thesis

$$\frac{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}{\vec{u} \triangleleft \mathfrak{B}} \text{ soundness} \qquad \frac{\vec{u} \triangleleft \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}} \text{ completeness?}$$

- ⋆ soundness is easy! Just count up the F-nodes.
- ⋆ completeness does not (generally) hold: procedure exists, but its termination requires Brouwer's Thesis!

Status of the Bar Thesis

- ⋆ Classically: valid

Status of the Bar Thesis

- ⋆ Classically: valid
- ⋆ Constructively: not valid

Status of the Bar Thesis

- ⋆ Classically: valid
- ⋆ Constructively: not valid
- ⋆ Intuitionistically: valid(*)

Let’s instantiate the Brouwer–Heyting–Kolmogorov interpretation at a simple theory of constructions!

System $\mathbb{T}$ as a theory of constructions

Gödel’s **System** $\mathbb{T}$ of primitive recursive functionals of finite type.

System $\mathbb{T}$ as a theory of constructions

Gödel's **System** $\mathbb{T}$ of primitive recursive functionals of finite type.

Atomic Types

$$\frac{}{\mathsf{nat}\ \mathit{atype}}$$

Types

$$\frac{\iota\ \mathit{atype}}{\iota\ \mathit{type}} \qquad \frac{\sigma\ \mathit{type} \quad \tau\ \mathit{type}}{\sigma \to \tau\ \mathit{type}}$$

Contexts

$$\frac{}{\cdot\ \mathit{ctx}} \qquad \frac{\Gamma\ \mathit{ctx} \quad \sigma\ \mathit{type}}{\Gamma, x : \sigma\ \mathit{ctx}}\ (x \notin \Gamma)$$

System 𝕋 as a theory of constructions

$$\frac{}{\Gamma, x : \sigma, \Delta \vdash x : \sigma}\ \mathsf{var}$$

$$\frac{}{\Gamma \vdash \mathsf{z} : \mathsf{nat}}\ \mathsf{zero} \qquad \frac{\Gamma \vdash m : \mathsf{nat}}{\Gamma \vdash \mathsf{s}(m) : \mathsf{nat}}\ \mathsf{succ}$$

$$\frac{\Gamma, x : \mathsf{nat}, y : \sigma \vdash s[x, y] : \sigma \quad \Gamma \vdash z : \sigma \quad \Gamma \vdash n : \mathsf{nat}}{\Gamma \vdash \mathsf{rec}_\sigma([x, y].s[x, y]; z; n) : \sigma}\ \mathsf{rec}$$

$$\frac{\Gamma, x : \sigma \vdash m[x] : \tau}{\Gamma \vdash \lambda x.m[x] : \sigma \to \tau}\ \mathsf{lam} \qquad \frac{\Gamma \vdash m : \sigma \to \tau \quad \Gamma \vdash n : \sigma}{\Gamma \vdash m \bullet_\sigma n : \tau}\ \mathsf{ap}$$

Realizability of barhood

- Use **System** $\mathbb{T}$ as a theory of constructions for primitive recursive arithmetic

Realizability of barhood

- ⋆ Use **System** $\mathbb{T}$ as a theory of constructions for primitive recursive arithmetic
- ⋆ $\vec{u} \lhd \mathfrak{B}$ should be realized by a functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$

Realizability of barhood

- ⋆ Use **System** $\mathbb{T}$ as a theory of constructions for primitive recursive arithmetic
- ⋆ $\vec{u} \lhd \mathcal{B}$ should be realized by a functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$
- ⋆ **Idea:** construct a model for **System** $\mathbb{T}$ in which we can read from the interpretation of ϕ a proof of $\vec{u} \blacktriangleleft_{ind} \mathcal{B}$

Standard semantics of System $\mathbb{T}$

Atomic Types

$$\mathcal{U}[\mathsf{nat}] \triangleq \mathbb{N}$$

Types

$$\mathcal{U}[\![\imath]\!] \triangleq \mathcal{U}[\imath]$$

$$\mathcal{U}[\![\sigma \to \tau]\!] \triangleq \mathcal{U}[\![\sigma]\!] \to \mathcal{U}[\![\tau]\!]$$

Contexts

$$\mathcal{G}[\![\Gamma]\!] \triangleq \prod_{x \in |\Gamma|} \mathcal{U}[\![\Gamma(x)]\!]$$

Standard semantics of System $\mathbb{T}$

Presupposing $\rho \in \mathcal{G}[\![\Gamma]\!]$ and $\Gamma \vdash m : \sigma$, define $[\![\Gamma \vdash m : \sigma]\!]_\rho \in \mathcal{U}[\![\sigma]\!]$ by recursion on m:

$$
\begin{aligned}
[\![\Gamma \vdash x : \sigma]\!]_\rho &\triangleq \rho(x) \\
[\![\Gamma \vdash \mathsf{z} : \mathsf{nat}]\!]_\rho &\triangleq 0 \\
[\![\Gamma \vdash \mathsf{s}(m) : \mathsf{nat}]\!]_\rho &\triangleq 1 + [\![\Gamma \vdash m : \mathsf{nat}]\!]_\rho \\
[\![\Gamma \vdash \mathsf{rec}_\sigma([x,y].s[x,y]; z; n) : \sigma]\!]_\rho &\triangleq \mathit{PrimRec}(S, Z, N)
\end{aligned}
$$

where

$$
\begin{aligned}
S(a,b) &\triangleq [\![\Gamma, x : \mathsf{nat}, y : \sigma \vdash s[x,y] : \sigma]\!]_{\rho, x \mapsto a, y \mapsto b} \\
Z &\triangleq [\![\Gamma \vdash z : \sigma]\!]_\rho \\
N &\triangleq [\![\Gamma \vdash n : \mathsf{nat}]\!]_\rho
\end{aligned}
$$

Standard semantics of System $\mathbb{T}$

$$\vdots$$

$$[\![\Gamma \vdash \lambda x.m[x] : \sigma \to \tau]\!]_\rho \triangleq a \mapsto [\![\Gamma, x : \sigma \vdash m[x] : \tau]\!]_{\rho, x \mapsto a}$$

$$[\![\Gamma \vdash m \bullet_\sigma n : \tau]\!]_\rho \triangleq \left([\![\Gamma \vdash m : \sigma \to \tau]\!]_\rho\right) \left([\![\Gamma \vdash n : \sigma]\!]_\rho\right)$$

Revising the Bar Thesis

A functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$ can be applied to a meta-level sequence α as follows:

$$\phi \langle \alpha \rangle \triangleq [\![\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}]\!] (\alpha)$$

Revising the Bar Thesis

A functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$ can be applied to a meta-level sequence α as follows:

$$\phi \langle \alpha \rangle \triangleq [\![\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}]\!] (\alpha)$$

Redefine barhood as follows:

$$\frac{\forall \alpha \in \vec{u}.\ \exists n \in \mathbb{N}.\ \overline{\alpha}[n] \in \mathfrak{B}}{\vec{u} \lhd \mathfrak{B}}$$

Revising the Bar Thesis

A functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$ can be applied to a meta-level sequence α as follows:

$$\phi\,\langle\alpha\rangle \triangleq [\![\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}]\!]\,(\alpha)$$

Redefine barhood as follows:

$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \vec{u}.\ \overline{\alpha}[\phi\,\langle\alpha\rangle] \in \mathfrak{B}}{\vec{u} \lhd_{\mathbb{T}} \mathfrak{B}}$$

Escardo dialogues

An inductive encoding of functionals $\mathbb{N}^{\mathbb{N}} \to Z$:

$$\frac{z \in Z}{\eta(z) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{return} \qquad \frac{x \in \mathbb{N} \quad e \in \mathbb{N} \to \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}{\varphi\langle x\rangle(e) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{query}$$

Escardo dialogues

An inductive encoding of functionals $\mathbb{N}^{\mathbb{N}} \to Z$:

$$\frac{z \in Z}{\eta(z) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{return} \qquad \frac{x \in \mathbb{N} \quad e \in \mathbb{N} \to \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}{\wp\langle x\rangle(e) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{query}$$

- ⋆ $\{\!\![\mathbb{N}^{\mathbb{N}}, -]\!\!\}$ is an endofunctor on **Set**

Escardo dialogues

An inductive encoding of functionals $\mathbb{N}^{\mathbb{N}} \to Z$:

$$\frac{z \in Z}{\eta(z) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \texttt{return} \qquad \frac{x \in \mathbb{N} \quad e \in \mathbb{N} \to \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}{\wp\langle x\rangle(e) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \texttt{query}$$

- ⋆ $\{\!\![\mathbb{N}^{\mathbb{N}}, -]\!\!\}$ is an endofunctor on **Set**
- ⋆ $\{\!\![\mathbb{N}^{\mathbb{N}}, -]\!\!\}$ is a monad on **Set**

Escardo dialogues

An inductive encoding of functionals $\mathbb{N}^{\mathbb{N}} \to Z$:

$$\frac{z \in Z}{\eta(z) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{return} \qquad \frac{x \in \mathbb{N} \quad e \in \mathbb{N} \to \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}{\varphi\langle x\rangle(e) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{query}$$

- ⋆ $\{\!\![\mathbb{N}^{\mathbb{N}}, -]\!\!\}$ is an endofunctor on **Set**
- ⋆ $\{\!\![\mathbb{N}^{\mathbb{N}}, -]\!\!\}$ is a monad on **Set**

souped up neighborhood functions!

Escardo dialogues

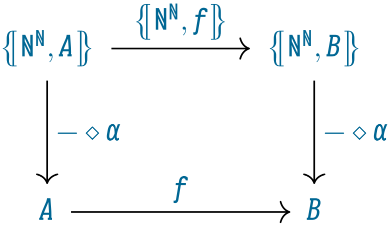
We can execute dialogue trees against a sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$:

$$\eta(z) \diamond \alpha \triangleq z$$
$$\varphi\langle x\rangle(e) \diamond \alpha \triangleq e(\alpha(x)) \diamond \alpha$$

Escardo dialogues

We can execute dialogue trees against a sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$:

$$\eta(z) \diamond \alpha \triangleq z$$
$$\varrho\langle x\rangle(e) \diamond \alpha \triangleq e(\alpha(x)) \diamond \alpha$$

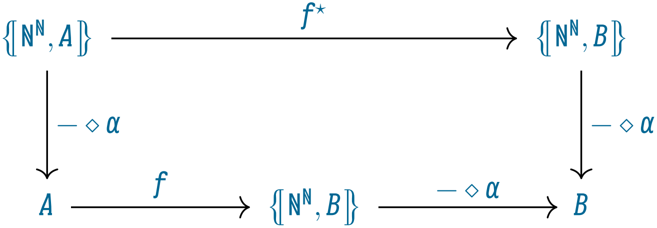


Escardo dialogues

We can execute dialogue trees against a sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$:

$$\eta(z) \diamond \alpha \triangleq z$$

$$\wp\langle x\rangle(e) \diamond \alpha \triangleq e(\alpha(x)) \diamond \alpha$$



Dialectical semantics of System 𝕋

Atomic Types

$$\mathcal{U}\,[\mathsf{nat}] \triangleq \mathbb{N}$$

Types

$$\mathcal{U}\,\langle\!\langle \iota \rangle\!\rangle \triangleq \{\!\!\{\mathbb{N}^{\mathbb{N}}, \mathcal{U}\,[\iota]\}\!\!\}$$

$$\mathcal{U}\,\langle\!\langle \sigma \to \tau \rangle\!\rangle \triangleq \mathcal{U}\,\langle\!\langle \sigma \rangle\!\rangle \to \mathcal{U}\,\langle\!\langle \tau \rangle\!\rangle$$

Contexts

$$\mathcal{G}\,\langle\!\langle \Gamma \rangle\!\rangle \triangleq \prod_{x \in |\Gamma|} \mathcal{U}\,\langle\!\langle \Gamma(x) \rangle\!\rangle$$

Dialectical semantics of System $\mathbb{T}$

Presupposing $\rho \in \mathcal{G}\langle\!\langle\Gamma\rangle\!\rangle$ and $\Gamma \vdash m : \sigma$, we define the interpretation $\langle\!\langle\Gamma \vdash m : \sigma\rangle\!\rangle_\rho \in \mathcal{U}\langle\!\langle\sigma\rangle\!\rangle$:

$$\langle\!\langle\Gamma \vdash x : \sigma\rangle\!\rangle_\rho \triangleq \rho(x)$$

$$\langle\!\langle\Gamma \vdash \mathsf{z} : \mathsf{nat}\rangle\!\rangle_\rho \triangleq \eta(0)$$

$$\langle\!\langle\Gamma \vdash \mathsf{s}(m) : \mathsf{nat}\rangle\!\rangle_\rho \triangleq \{\!\![\mathsf{N}^\mathsf{N}, 1 + -]\!\!\}\left(\langle\!\langle\Gamma \vdash m : \mathsf{nat}\rangle\!\rangle_\rho\right)$$

$$\langle\!\langle\Gamma \vdash \mathsf{rec}_\sigma([x,y].s[x,y]; z; n) : \sigma\rangle\!\rangle_\rho \triangleq \mathit{PrimRec}(S, Z, -)^{\circledast}_{\sigma}(N)$$

where

$$S(a,b) \triangleq \langle\!\langle\Gamma, x : \mathsf{nat}, y : \sigma \vdash s[x,y] : \sigma\rangle\!\rangle_{\rho, x\mapsto a, y\mapsto b}$$

$$Z \triangleq \langle\!\langle\Gamma \vdash z : \sigma\rangle\!\rangle_\rho$$

$$N \triangleq \langle\!\langle\Gamma \vdash n : \mathsf{nat}\rangle\!\rangle_\rho$$

Dialectical semantics of System $\mathbb{T}$

$$\vdots$$

$$\langle\!\langle \Gamma \vdash \lambda x.m[x] : \sigma \to \tau \rangle\!\rangle_\rho \triangleq a \mapsto \langle\!\langle \Gamma, x : \sigma \vdash m[x] : \tau \rangle\!\rangle_{\rho, x \mapsto a}$$

$$\langle\!\langle \Gamma \vdash m \bullet_\sigma n : \tau \rangle\!\rangle_\rho \triangleq \left(\langle\!\langle \Gamma \vdash m : \sigma \to \tau \rangle\!\rangle_\rho \right) \left(\langle\!\langle \Gamma \vdash n : \sigma \rangle\!\rangle_\rho \right)$$

Coherence via Logical Relations

We need to show that the two interpretations cohere at base type, i.e. that the following diagram commutes:

$$\begin{array}{ccc} & \cdot \vdash \iota & \\ {\scriptstyle \langle\!\langle \cdot \vdash - : \iota \rangle\!\rangle}\swarrow & & \searrow{\scriptstyle [\![\cdot \vdash - : \iota]\!]} \\ U\,\langle\!\langle \iota \rangle\!\rangle & \xrightarrow[- \diamond \alpha]{\qquad\qquad} & U\,[\![\iota]\!] \end{array}$$

Coherence via Logical Relations

We need to show that the two interpretations cohere at base type, i.e. that the following diagram commutes:

$$\begin{array}{ccc} & \cdot \vdash \iota & \\ \langle\!\langle \cdot \vdash - : \iota \rangle\!\rangle \swarrow & & \searrow [\![\cdot \vdash - : \iota]\!] \\ U \langle\!\langle \iota \rangle\!\rangle & \xrightarrow[\; - \diamond \alpha \;]{} & U [\![\iota]\!] \end{array}$$

SLOGAN: STRENGTHEN THE INDUCTIVE HYPOTHESIS WITH LOGICAL RELATIONS AS A WEAPON!

Coherence via Logical Relations

"logical relations" = family of relations defined by compositionally on object-level types

Coherence via Logical Relations

"logical relations" = family of relations defined by compositionally on object-level types

For all $\alpha \in \mathbb{N}^{\mathbb{N}}$, define:

$$\mathcal{R}^{\alpha}_{\sigma} \subseteq \mathcal{U}\,[\![\sigma]\!] \times \mathcal{U}\,\langle\!\langle\sigma\rangle\!\rangle \qquad \overline{\mathcal{R}^{\alpha}_{\Gamma}} \subseteq \mathcal{G}\,[\![\Gamma]\!] \times \mathcal{G}\,\langle\!\langle\Gamma\rangle\!\rangle$$

$$\frac{F = d \diamond \alpha}{F\ \mathcal{R}^{\alpha}_{\iota}\ d} \qquad \frac{\forall G \in \mathcal{U}\,[\![\sigma]\!], e \in \mathcal{U}\,\langle\!\langle\sigma\rangle\!\rangle.\ G\ \mathcal{R}^{\alpha}_{\sigma}\ e \implies F(G)\ \mathcal{R}^{\alpha}_{\tau}\ d(e)}{F\ \mathcal{R}^{\alpha}_{\sigma\to\tau}\ d}$$

$$\frac{\forall x \in |\Gamma|.\ \rho_0(x)\ \mathcal{R}^{\alpha}_{\Gamma(x)}\ \rho_1(x)}{\rho_0\ \overline{\mathcal{R}^{\alpha}_{\Gamma}}\ \rho_1}$$

The Generic Point

In the dialogue model, we can define a so-called “generic point” which is not definable in **System T**:

$$\mathit{generic} \in \mathcal{U}\,\langle\!\langle \mathsf{nat} \to \mathsf{nat} \rangle\!\rangle$$

$$\mathit{generic} \triangleq (\wp\langle - \rangle(\eta))^{\star}$$

The Generic Point

In the dialogue model, we can define a so-called “generic point” which is not definable in **System $\mathbb{T}$**:

$$\mathit{generic} \in \mathcal{U}\langle\!\langle \mathsf{nat} \to \mathsf{nat} \rangle\!\rangle$$

$$\mathit{generic} \triangleq (\varphi\langle - \rangle(\eta))^{\star}$$

THE GENERIC POINT IS THE MAGIC WEAPON TO VICTORIOUSLY TRACE A FUNCTIONAL'S INTERACTION WITH THE AMBIENT CHOICE SEQUENCE!

— **Quotations From Chairman Thierry Coquand**

Taking stock

- For any functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$, we can compute a dialogue tree $\langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\, (\mathit{generic}) \in \{\![\mathbb{N}^{\mathbb{N}}, \mathbb{N}]\!\}$.

Taking stock

- ⋆ For any functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$, we can compute a dialogue tree $\langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\,(\mathit{generic}) \in \{\![\mathbb{N}^{\mathbb{N}}, \mathbb{N}]\!\}$.
- ⋆ Similar to a Brouwerian mental construction of $\vec{u} \blacktriangleleft_{ind} \mathfrak{B}$, but not the same!

Taking stock

- For any functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$, we can compute a dialogue tree $\langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\,(\mathit{generic}) \in \{\![\mathbb{N}^{\mathbb{N}}, \mathbb{N}]\!\}$.
- Similar to a Brouwerian mental construction of $\vec{u} \blacktriangleleft_{ind} \mathbb{B}$, but not the same!
- Escardó's trees represent ***persistent inspection*** of a choice sequence; Brouwer's trees represent ***ephemeral consumption*** of a choice sequence.

Taking stock

- For any functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$, we can compute a dialogue tree $\langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\,(\mathit{generic}) \in \{\![\mathbb{N}^{\mathbb{N}}, \mathbb{N}]\!\}$.
- Similar to a Brouwerian mental construction of $\vec{u} \blacktriangleleft_{ind} \mathfrak{B}$, but not the same!
- Escardó's trees represent ***persistent inspection*** of a choice sequence; Brouwer's trees represent ***ephemeral consumption*** of a choice sequence.
- **Idea:** Normalize dialogue trees into Brouwerian mental constructions.

Brouwer's ephemeral dialectics

Recall Escardó's dialogues:

$$\frac{z \in Z}{\eta(z) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{return} \qquad \frac{x \in \mathbb{N} \quad e \in \mathbb{N} \to \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}{\varphi\langle x\rangle(e) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{query}$$

Brouwer's ephemeral dialectics

Recall Escardó's dialogues:

$$\frac{z \in Z}{\eta(z) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{return} \qquad \frac{x \in \mathbb{N} \quad e \in \mathbb{N} \to \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}{\varphi\langle x\rangle(e) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{query}$$

Brouwer's take:

$$\frac{z \in Z}{\eta(z) \in (\!|\mathbb{N}^{\mathbb{N}}, Z|\!)}\ \text{spit} \qquad \frac{b \in \mathbb{N} \to (\!|\mathbb{N}^{\mathbb{N}}, Z|\!)}{\mathsf{F}(b) \in (\!|\mathbb{N}^{\mathbb{N}}, Z|\!)}\ \text{bite}$$

Brouwer's ephemeral dialectics

Recall Escardó's dialogues:

$$\frac{z \in Z}{\eta(z) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{return} \qquad \frac{x \in \mathbb{N} \quad e \in \mathbb{N} \to \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}{\varrho\langle x\rangle(e) \in \{\!\![\mathbb{N}^{\mathbb{N}}, Z]\!\!\}}\ \text{query}$$

Brouwer's take:

$$\frac{z \in Z}{\eta(z) \in (\!|\mathbb{N}^{\mathbb{N}}, Z|\!)}\ \text{spit} \qquad \frac{b \in \mathbb{N} \to (\!|\mathbb{N}^{\mathbb{N}}, Z|\!)}{\mathsf{F}(b) \in (\!|\mathbb{N}^{\mathbb{N}}, Z|\!)}\ \text{bite}$$

Ephemeral execution:

$$\eta(z) \diamond \alpha \triangleq z$$
$$\mathsf{F}(b) \diamond \alpha \triangleq b(\mathsf{head}(\alpha)) \diamond \mathsf{tail}(\alpha)$$

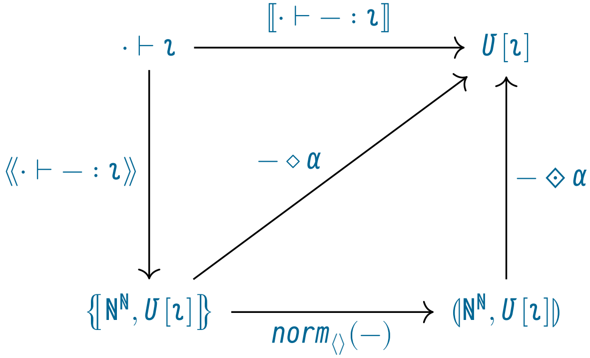
Normalizing dialogues

Presupposing $t \in \{\!|\mathbb{N}^{\mathbb{N}}, Z|\!\}$, define total normalization relation $\vec{u} \Vdash t \rightsquigarrow b$ with $b \in (\!|\mathbb{N}^{\mathbb{N}}, Z|\!)$. Then, define,

$$\frac{\vec{u} \Vdash t \rightsquigarrow b}{norm_{\vec{u}}(t) \triangleq b}$$

Structurally recursive definition easy, but bureaucratic. See paper.

The Birds'-Eye View



The Bar Thesis

The Bar Theorem

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\vec{u} \lhd_{\mathbb{T}} \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}} \text{ completeness}$$

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \vec{u}.\ \overline{\alpha}[\phi\,\langle \alpha \rangle] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\exists \cdot \vdash \phi : (\mathtt{nat} \to \mathtt{nat}) \to \mathtt{nat}.\ \forall \alpha \in \langle\rangle.\ \overline{\vec{u} \oplus \alpha}[\phi\ \langle\alpha\rangle + |\vec{u}|] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \langle\rangle.\ \overline{\vec{u} \oplus \alpha}[\phi\ \langle\alpha\rangle + |\vec{u}|] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

Proof. Let $t \triangleq \langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\ (\mathit{generic})$ such that $\langle\rangle \Vdash t \rightsquigarrow b$.

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \langle\rangle.\ \overline{\vec{u} \oplus \alpha}[\phi\ \langle\alpha\rangle + |\vec{u}|] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

Proof. Let $t \triangleq \langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\ (\mathit{generic})$ such that $\langle\rangle \Vdash t \rightsquigarrow b$. By coherence, we have $\overline{\vec{u} \oplus \alpha}[b \diamond \alpha + |\vec{u}|] \in \mathfrak{B}$ for all α.

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \langle\rangle.\ \overline{\vec{u} \oplus \alpha}[\phi\ \langle\alpha\rangle + |\vec{u}|] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

Proof. Let $t \triangleq \langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\ (\mathit{generic})$ such that $\langle\rangle \Vdash t \rightsquigarrow b$. By coherence, we have $\overline{\vec{u} \oplus \alpha}[b \diamond \alpha + |\vec{u}|] \in \mathfrak{B}$ for all α. Proceed by induction on $b \in (\!| \mathbb{N}^{\mathbb{N}}, \mathbb{N} |\!)$.

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \langle\rangle.\ \overline{\vec{u} \oplus \alpha}[\phi\ \langle\alpha\rangle + |\vec{u}|] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

Proof. Let $t \triangleq \langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\ (\mathit{generic})$ such that $\langle\rangle \Vdash t \rightsquigarrow b$. By coherence, we have $\overline{\vec{u} \oplus \alpha}[b \diamond \alpha + |\vec{u}|] \in \mathfrak{B}$ for all α. Proceed by induction on $b \in (\!| \mathbb{N}^{\mathbb{N}}, \mathbb{N} |\!)$.

- ⋆ *Case* $b \equiv \eta(0)$. Then we already have $\vec{u} \in \mathfrak{B}$, so apply η rule.

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \langle\rangle.\ \overline{\vec{u} \oplus \alpha}[\phi\ \langle\alpha\rangle + |\vec{u}|] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

Proof. Let $t \triangleq \langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle$ $(generic)$ such that $\langle\rangle \Vdash t \rightsquigarrow b$. By coherence, we have $\overline{\vec{u} \oplus \alpha}[b \diamond \alpha + |\vec{u}|] \in \mathfrak{B}$ for all α. Proceed by induction on $b \in (\!|\mathbb{N}^{\mathbb{N}}, \mathbb{N}|\!)$.

- ⋆ *Case* $b \equiv \eta(0)$. Then we already have $\vec{u} \in \mathfrak{B}$, so apply η rule.
- ⋆ *Case* $b \equiv \eta(k + 1)$. Apply $\digamma$ rule, and use IH at $\eta(k)$.

The Bar Theorem

It suffices to show the validity of the following completeness rule:

$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \langle\rangle.\ \overline{\vec{u} \oplus \alpha}[\phi\,\langle\alpha\rangle + |\vec{u}|] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

Proof. Let $t \triangleq \langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle$ $(\textit{generic})$ such that $\langle\rangle \Vdash t \rightsquigarrow b$. By coherence, we have $\overline{\vec{u} \oplus \alpha}[b \diamond \alpha + |\vec{u}|] \in \mathfrak{B}$ for all α. Proceed by induction on $b \in (\!|\mathbb{N}^{\mathbb{N}}, \mathbb{N}|\!)$.

- ⋆ *Case* $b \equiv \eta(0)$. Then we already have $\vec{u} \in \mathfrak{B}$, so apply η rule.
- ⋆ *Case* $b \equiv \eta(k+1)$. Apply F rule, and use IH at $\eta(k)$.
- ⋆ *Case* $b \equiv \mathsf{F}(f)$. Apply F rule, fixing $y \in \mathbb{N}$; use IH at $f(y)$.

The Bar Theorem

It suffices to show the validity of the following completeness rule:

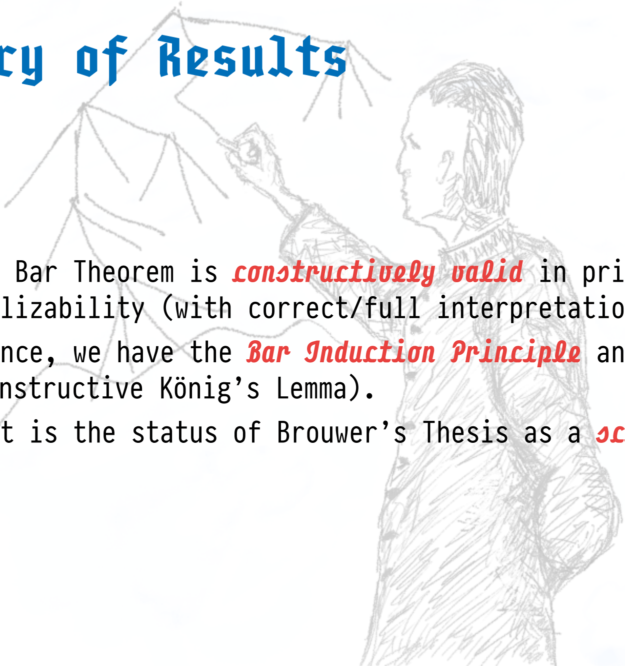
$$\frac{\exists \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \in \langle\rangle.\ \overline{\vec{u} \oplus \alpha}[\phi\,\langle\alpha\rangle + |\vec{u}|] \in \mathfrak{B}}{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}\ \text{completeness}$$

Proof. Let $t \triangleq \langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\ (\mathit{generic})$ such that $\langle\rangle \Vdash t \rightsquigarrow b$. By coherence, we have $\overline{\vec{u} \oplus \alpha}[b \diamond \alpha + |\vec{u}|] \in \mathfrak{B}$ for all α. Proceed by induction on $b \in (\!|\mathbb{N}^{\mathbb{N}}, \mathbb{N}|\!)$.

- ⋆ *Case* $b \equiv \eta(0)$. Then we already have $\vec{u} \in \mathfrak{B}$, so apply η rule.
- ⋆ *Case* $b \equiv \eta(k+1)$. Apply F rule, and use IH at $\eta(k)$.
- ⋆ *Case* $b \equiv \mathsf{F}(f)$. Apply F rule, fixing $y \in \mathbb{N}$; use IH at $f(y)$.

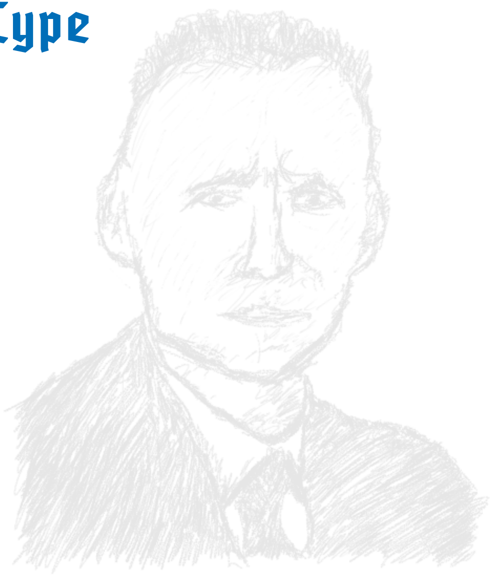
□

Summary of Results



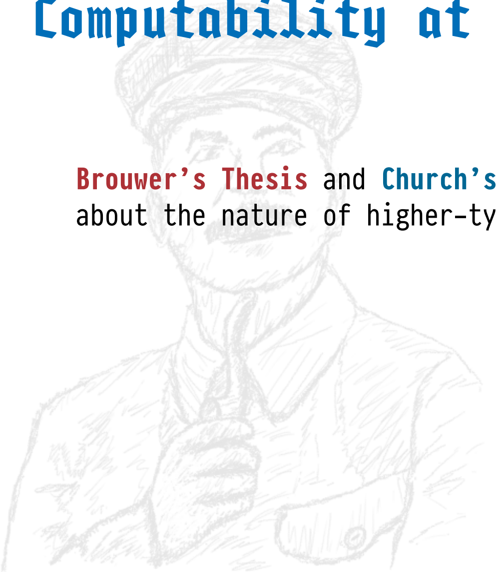
- The Bar Theorem is ***constructively valid*** in primitive recursive realizability (with correct/full interpretation of functional types).
- Thence, we have the ***Bar Induction Principle*** and the ***Fan Theorem*** (constructive König's Lemma).
- What is the status of Brouwer's Thesis as a ***scientific hypothesis***?

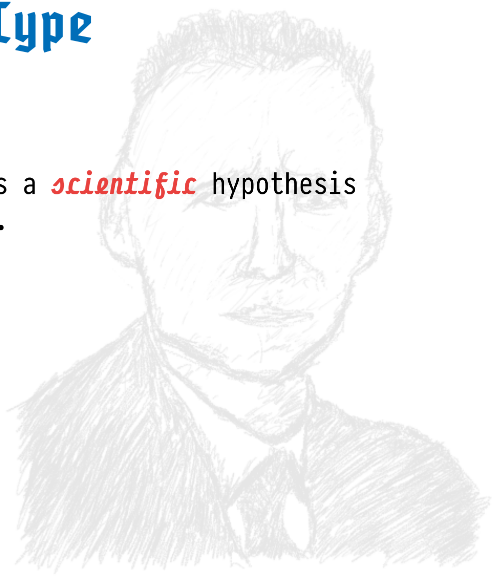
Computability at Higher Type



vulgar mechanicalism vs. phenomenology

Computability at Higher Type





Brouwer's Thesis and **Church's Thesis** express a *scientific* hypothesis about the nature of higher-type computation.

Computability at Higher Type

Brouwer's Thesis and **Church's Thesis** express a *scientific* hypothesis about the nature of higher-type computation.

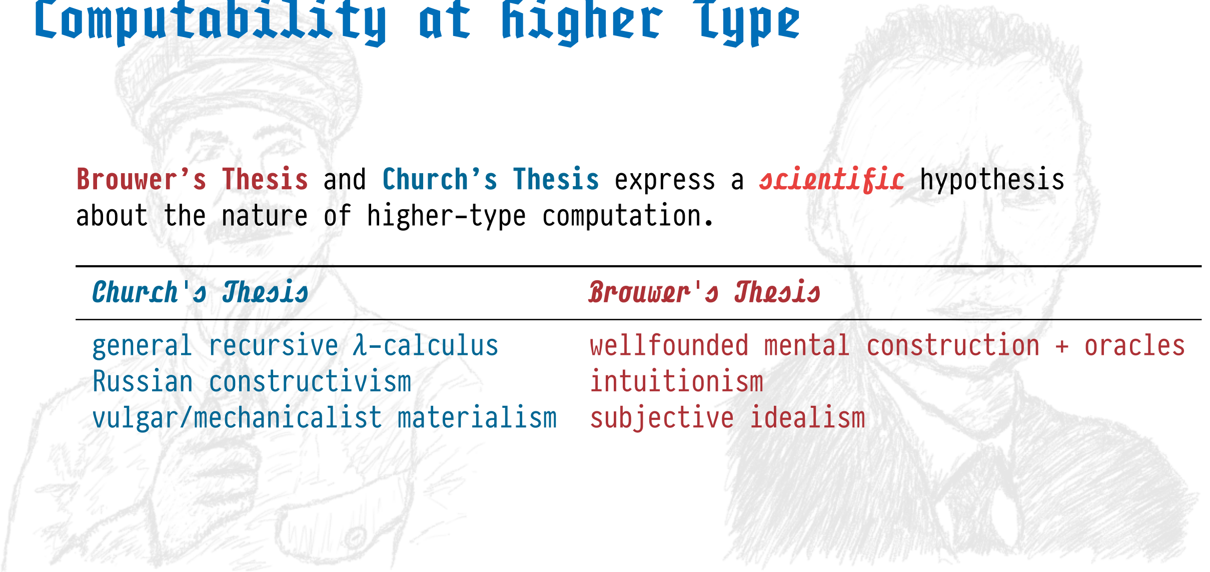
Church's Thesis	*Brouwer's Thesis*
general recursive λ-calculus	wellfounded mental construction + oracles

Computability at Higher Type

Brouwer's Thesis and **Church's Thesis** express a *scientific* hypothesis about the nature of higher-type computation.

Church's Thesis	*Brouwer's Thesis*
general recursive λ-calculus Russian constructivism	wellfounded mental construction + oracles intuitionism

Computability at Higher Type



Brouwer’s Thesis and **Church’s Thesis** express a *scientific* hypothesis about the nature of higher-type computation.

Church's Thesis	*Brouwer's Thesis*
general recursive λ-calculus	wellfounded mental construction + oracles
Russian constructivism	intuitionism
vulgar/mechanicalist materialism	subjective idealism

Computability at Higher Type

Brouwer's Thesis and **Church's Thesis** express a *scientific* hypothesis about the nature of higher-type computation.

Church's Thesis	*Brouwer's Thesis*
general recursive λ-calculus	wellfounded mental construction + oracles
Russian constructivism	intuitionism
vulgar/mechanicalist materialism	subjective idealism
uniform continuity fails on $\mathbb{I}$	uniform continuity obtains