

Algebraic Type Theory and Universe Hierarchies

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Abstract

It is commonly believed that algebraic notions of type theory support only universes à la Tarski, and that universes à la Russell must be removed by elaboration. We clarify the state of affairs, recalling the details of Cartmell’s discipline of *generalized algebraic theory* [Car78], showing how to formulate an algebraic version of Coquand’s *cumulative cwfs* with universes à la Russell.

To demonstrate the power of algebraic techniques, we sketch a purely algebraic proof of canonicity for Martin-Löf Type Theory with universes, dependent function types, and a base type with two constants.

1 Generalized algebraic theories

Cartmell [Car78] defines a notion of *generalized algebraic theory*, generated by a collection of *formation rules* for sort symbols, *introduction rules* for operation symbols, and axioms for both sort equality and term equality.¹ These are all inter-dependent, so care must be taken to stage the construction properly. In this note, we give a more streamlined presentation with a few differences from Cartmell’s:

1. At a superficial level, we use a modernized notation, inspired by logical frameworks.
2. At a technical level, we stage the construction in such a way that, when considering the *derived rules* available for a signature, we have already assumed that the signature is well-formed.

It’s worth noting that our formulation, while suitable for the development of signatures which are finitary, is less general than the notion considered by Cartmell; our version does not suffice to develop the universal algebra of generalized algebraic theories (including the equivalence between the category of such theories and the category of contextual categories), as noted by Taylor [Tay99].

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¹Taylor [Tay99] considers a version of generalized algebraic theory which omits generating sort equations (sort equality must still be considered as a consequence of sorts depending on terms).

1.1 Grammar and substitution

In this section, we present an informal grammar to generate the raw syntax whose well-formedness we characterize in [Section 1.3](#).

(judgments)	\mathcal{J}	$::=$	$\mathbb{T} \text{thy} \mid \Psi \text{tele}_{\mathbb{T}} \mid \Psi \vdash_{\mathbb{T}} A \text{sort} \mid \Psi \vdash_{\mathbb{T}} M : A \mid \Phi \vdash_{\mathbb{T}} \psi : \Psi$ $\Psi \vdash_{\mathbb{T}} A = B \text{sort} \mid \Psi \vdash_{\mathbb{T}} M = N : A \mid \Phi \vdash_{\mathbb{T}} \psi_0 = \psi_1 : \Psi$
(theories)	\mathbb{T}	$::=$	$\emptyset \mid \mathbb{T}, \vartheta : \mathcal{D} \mid \mathbb{T}, \mathcal{E}$
(declarations)	\mathcal{D}	$::=$	$[\Psi \vdash \text{sort}] \mid [\Psi \vdash A]$
(axioms)	\mathcal{E}	$::=$	$[\Psi \vdash A = B \text{sort}] \mid [\Psi \vdash M = N : A]$
(telescopes)	Ψ	$::=$	$\cdot \mid \Psi, x : A$
(sorts)	A	$::=$	$\vartheta\langle\psi\rangle$
(terms)	M	$::=$	$x \mid \vartheta\langle\psi\rangle$
(substitutions)	ψ	$::=$	$\cdot \mid \psi, M/x$

We define the action of raw substitutions on raw sorts and terms $\psi^*(-)$ and the composition of raw substitutions $\psi \circ \phi$ by recursion as follows:

$$\begin{aligned}
 \cdot * x &= x \\
 (\psi, M/x)^* x &= x & \cdot \circ \psi &= \cdot \\
 (\psi, M/y)^* x &= \psi^* x & (\phi, N/x) \circ \psi &= (\phi \circ \psi), (\psi^* N)/x \\
 \psi^* \vartheta\langle\phi\rangle &= \vartheta\langle\phi \circ \psi\rangle
 \end{aligned}$$

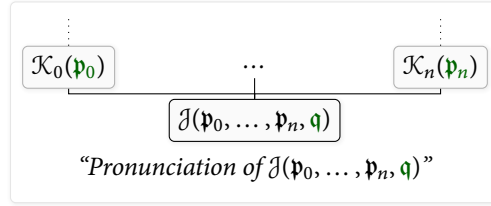
1.2 Judgments and presuppositions

We will define several forms of judgment simultaneously, to specify the well-formedness and equality conditions for theories, sorts, terms and substitutions. Following Martin-Löf [Mar96] and Schroeder-Heister [Sch87], we explain a form of judgment by *first* specifying its presuppositions (what are the meaningful instances of the form of judgment?), and then giving rules which specify when a meaningful instance of a form of judgment can be verified.

For instance, to specify a form of judgment like “ M is of sort A ”, we first *presuppose* that A is already known to be a sort, but require of M only that it is generated from an appropriate production of the raw syntax in [Section 1.1](#); then, rather than being false, the spurious instance “ M is of sort 57” is actually not assigned a meaning at all. The discipline of presuppositions enables us to omit many redundant premises from rules.

Convention 1.1 (Subjects and presuppositions). *In all cases that we will consider here, a form of judgment expresses that some piece of raw syntax (the **subject**) is well-formed relative to some other objects which are already presupposed to be well-formed (the **parameters**); we write the subject in color to distinguish it visually from the parameters. We indicate this*

situation schematically for a form of judgment \mathcal{J} in the following way:



The above schema should be read as asserting that the judgment $\mathcal{J}(p_0, \dots, p_n, q)$ presupposes $\mathcal{K}_0(p_0)$ through $\mathcal{K}_n(p_n)$, transitively presupposing whatever is presupposed by $\mathcal{K}_i(p_i)$, establishing the well-formedness of the raw syntax q .

Once the meaningful instances of a form of judgment have been generated schematically as in **Convention 1.1**, its *correct* instances can be characterized inductively by rules of inference.

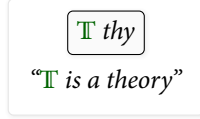
Relation to traditional presentations The “traditional” presentation of rules of inference, in which *all* constituents are treated as subjects and inference rules are equipped with extra premises to govern their well-formedness, can be obtained in a completely mechanical way from the more streamlined systems presented here.

A form of judgment $\mathcal{J}(\vec{p}, q)$ should be thought of as a family of sets of derivations indexed in $\{(\vec{p}, q) \mid \overline{\mathcal{K}(\vec{p})}\}$. The act of forgetting the well-formedness of the parameters induces a contravariant restriction of forms of judgment in an adjoint situation:

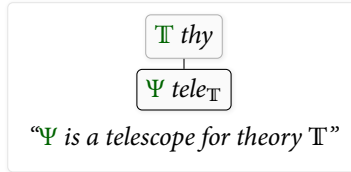
$$\begin{array}{ccc}
 \{(\vec{p}, q) \mid \overline{\mathcal{K}(\vec{p})}\} & \mathbf{Jdg}(\{(\vec{p}, q)\}) \\
 \downarrow i & \begin{array}{c} \uparrow \exists_i \\ \dashv i^* \\ \downarrow \forall_i \end{array} \\
 \{(\vec{p}, q)\} & \mathbf{Jdg}(\{(\vec{p}, q) \mid \overline{\mathcal{K}(\vec{p})}\})
 \end{array}$$

From the left adjoint \exists_i , we obtain exactly the system of judgments and rules without presuppositions, in which the old presuppositions are added as auxiliary premises to every rule in just the right place.

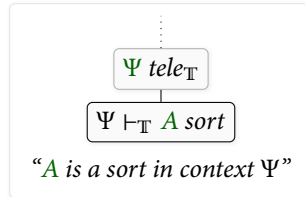
1.3 Rules for generalized algebraic theories



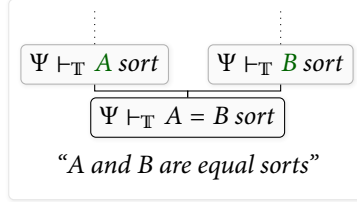
EMPTY	$\frac{\mathbb{T} \text{ thy}}{\emptyset \text{ thy}}$	$\frac{\text{SORT DECL} \quad \mathbb{T} \text{ thy} \quad \Psi \text{ tele}_{\mathbb{T}} \quad (\vartheta \notin \mathbb{T})}{\mathbb{T}, \vartheta : [\Psi \vdash \text{sort}] \text{ thy}}$	$\frac{\text{OPERATION DECL} \quad \mathbb{T} \text{ thy} \quad \Psi \text{ tele}_{\mathbb{T}} \quad \Psi \vdash_{\mathbb{T}} A \text{ sort} \quad (\vartheta \notin \mathbb{T})}{\mathbb{T}, \vartheta : [\Psi \vdash A] \text{ thy}}$
		$\frac{\text{SORT AXIOM} \quad \mathbb{T} \text{ thy} \quad \Psi \text{ tele}_{\mathbb{T}} \quad \Psi \vdash_{\mathbb{T}} A_0 \text{ sort} \quad \Psi \vdash_{\mathbb{T}} A_1 \text{ sort}}{\mathbb{T}, [\Psi \vdash A_0 = A_1 \text{ sort}] \text{ thy}}$	
		$\frac{\text{TERM AXIOM} \quad \mathbb{T} \text{ thy} \quad \Psi \text{ tele}_{\mathbb{T}} \quad \Psi \vdash_{\mathbb{T}} A \text{ sort} \quad \Psi \vdash_{\mathbb{T}} M_0 : A \quad \Psi \vdash_{\mathbb{T}} M_1 : A}{\mathbb{T}, [\Psi \vdash M_0 = M_1 : A] \text{ thy}}$	



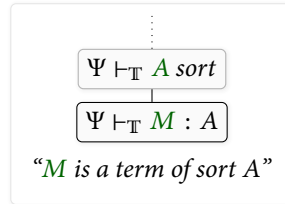
$\frac{\text{EMPTY}}{\cdot \text{tele}_{\mathbb{T}}}$	$\frac{\text{SNOC} \quad \Psi \text{ tele}_{\mathbb{T}} \quad \Psi \vdash_{\mathbb{T}} A \text{ sort} \quad (x \notin \Psi)}{\Psi, x : A \text{ tele}_{\mathbb{T}}}$
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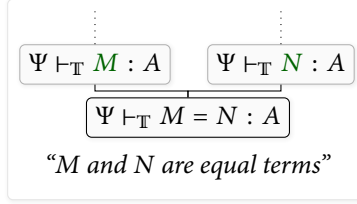
$\frac{\text{SORT FORMATION} \quad \mathbb{T} \ni \vartheta : [\Phi \vdash \text{sort}] \quad \Psi \vdash_{\mathbb{T}} \varphi : \Phi}{\Psi \vdash_{\mathbb{T}} \vartheta(\varphi) \text{ sort}}$



<p>REFLEXIVITY</p> $\frac{}{\Psi \vdash_{\mathbb{T}} A = A \text{ sort}}$	<p>SYMMETRY</p> $\frac{\Psi \vdash_{\mathbb{T}} A_0 = A_1 \text{ sort}}{\Psi \vdash_{\mathbb{T}} A_1 = A_0 \text{ sort}}$	<p>TRANSITIVITY</p> $\frac{\Psi \vdash_{\mathbb{T}} A_0 = A_1 \text{ sort} \quad \Psi \vdash_{\mathbb{T}} A_1 = A_2 \text{ sort}}{\Psi \vdash_{\mathbb{T}} A_0 = A_2 \text{ sort}}$
<p>SUBSTITUTION</p> $\frac{\Phi \vdash_{\mathbb{T}} A_0 = A_1 \text{ sort} \quad \Psi \vdash_{\mathbb{T}} \phi_0 = \phi_1 : \Phi}{\Psi \vdash_{\mathbb{T}} \phi_0^* A_0 = \phi_1^* A_1 \text{ sort}}$		<p>SORT AXIOM</p> $\frac{\mathbb{T} \ni [\Psi \vdash A_0 = A_1 \text{ sort}]}{\Psi \vdash_{\mathbb{T}} A_0 = A_1 \text{ sort}}$



<p>TERM FORMATION</p> $\frac{\mathbb{T} \ni \vartheta : [\Phi \vdash A] \quad \Psi \vdash_{\mathbb{T}} \varphi : \Phi}{\Psi \vdash_{\mathbb{T}} \vartheta\langle\varphi\rangle : \phi^* A}$	<p>VARIABLE</p> $\frac{}{\Psi, x : A \vdash_{\mathbb{T}} x : A}$
<p>CONVERSION</p> $\frac{\Psi \vdash_{\mathbb{T}} A_0 = A_1 \text{ sort} \quad \Psi \vdash_{\mathbb{T}} M : A_0}{\Psi \vdash_{\mathbb{T}} M : A_1}$	



REFLEXIVITY

$$\frac{}{\Psi \vdash_{\mathbb{T}} M = M : A}$$

SYMMETRY

$$\frac{\Psi \vdash_{\mathbb{T}} M_0 = M_1 : A}{\Psi \vdash_{\mathbb{T}} M_1 = M_0 : A}$$

TRANSITIVITY

$$\frac{\Psi \vdash_{\mathbb{T}} M_0 = M_1 : A \quad \Psi \vdash_{\mathbb{T}} M_1 = M_2 : A}{\Psi \vdash_{\mathbb{T}} M_0 = M_2 : A}$$

CONVERSION

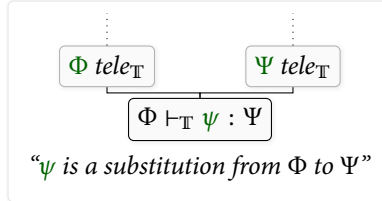
$$\frac{\Psi \vdash_{\mathbb{T}} M_0 = M_1 : A_0 \quad \Psi \vdash_{\mathbb{T}} A_0 = A_1 \text{ sort}}{\Psi \vdash_{\mathbb{T}} M_0 = M_1 : A_1}$$

SUBSTITUTION

$$\frac{\Phi \vdash_{\mathbb{T}} M_0 = M_1 : A \quad \Psi \vdash_{\mathbb{T}} \phi_0 = \phi_1 : \Phi}{\Psi \vdash_{\mathbb{T}} \phi_0^* M_0 = \phi_1^* M_1 : \phi_0^* A}$$

TERM AXIOM

$$\frac{\mathbb{T} \ni [\Psi \vdash M_0 = M_1 : A]}{\Psi \vdash_{\mathbb{T}} M_0 = M_1 : A}$$

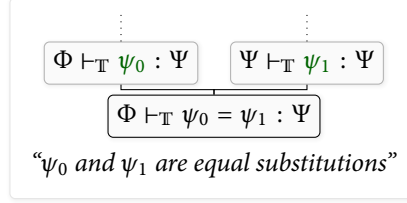


EMPTY

$$\frac{}{\Phi \vdash_{\mathbb{T}} \dots}$$

SNOC

$$\frac{\Phi \vdash_{\mathbb{T}} \psi : \Psi \quad \Phi \vdash_{\mathbb{T}} M : \psi^* A}{\Phi \vdash_{\mathbb{T}} (\psi, M/x) : \Psi, x : A}$$



$$\frac{\text{EMPTY}}{\Phi \vdash_{\mathbb{T}} \cdot \dots \cdot} \quad \frac{\text{SNOC} \quad \Phi \vdash_{\mathbb{T}} \psi_0 = \psi_1 : \Psi \quad \Phi \vdash_{\mathbb{T}} M_0 = M_1 : A[\psi_0]}{\Phi \vdash_{\mathbb{T}} (\psi_0, M_0/x) = (\psi_1, M_1/x) : \Psi, x : A}$$

Notation 1.2 (Substitution). Abusing notation, we will often write ψ, M instead of $\psi, M/x$ when the variable is clear from context; we will also routinely write M instead of \cdot, M .

Lemma 1.3 (Substitution). *When $\Phi \vdash_{\mathbb{T}} \psi : \Psi$, we have the following admissible substitution principles:*

1. *If $\Psi \vdash_{\mathbb{T}} A$ sort, then $\Phi \vdash_{\mathbb{T}} \psi^* A$ sort.*
2. *If $\Psi \vdash_{\mathbb{T}} M : A$, then $\Phi \vdash_{\mathbb{T}} \psi^* M : \psi^* A$.*
3. *If $\Psi \vdash_{\mathbb{T}} \xi : \Xi$, then $\Phi \vdash_{\mathbb{T}} \xi \circ \psi : \Xi$.*

Proof. By mutual induction on derivations.

1.4 Notation for theories

Based on the rules and grammar that we have given, a generalized algebraic theory is defined by a sequence of declarations of sort symbols and operation symbols and their arities, with equational axioms interspersed, subject to the sorting discipline. For instance, the generalized algebraic theory of monoids is given as follows:

$$\begin{aligned} \mathbf{ob} &: [\cdot \vdash \text{sort}], \\ \mathbf{id} &: [\cdot \vdash \mathbf{ob}\langle \cdot \rangle], \\ \mathbf{op} &: [x : \mathbf{ob}\langle \cdot \rangle, y : \mathbf{ob}\langle \cdot \rangle \vdash \mathbf{ob}\langle \cdot \rangle], \\ [x : \mathbf{ob}\langle \cdot \rangle \vdash \mathbf{op}\langle x, \mathbf{id}\langle \cdot \rangle \rangle &= x : \mathbf{ob}\langle \cdot \rangle], \\ [x : \mathbf{ob}\langle \cdot \rangle \vdash \mathbf{op}\langle \mathbf{id}\langle \cdot \rangle, x \rangle &= x : \mathbf{ob}\langle \cdot \rangle], \\ [x : \mathbf{ob}\langle \cdot \rangle, y : \mathbf{ob}\langle \cdot \rangle, z : \mathbf{ob}\langle \cdot \rangle \vdash \mathbf{op}\langle \mathbf{op}\langle x, y \rangle, z \rangle &= \mathbf{op}\langle x, \mathbf{op}\langle y, z \rangle \rangle : \mathbf{ob}\langle \cdot \rangle] \end{aligned}$$

However, for more complex theories, this linear notation will be a hindrance; therefore, we will impose an inference-style notation which will be more ergonomic. In our informal notation, a formation rule for a sort or operation symbol will simultaneously extend the signature with the appropriate declaration, and impose an informal *notation* for its use. It is crucial to note that these notations are just that: they are not part of the formal theory, but instead part of the informal way that we render the real objects of the theory (concrete trees) into linear text.

Sort declaration The sort formation rule $\frac{x_0 : A_0 \quad \dots \quad x_n : A_n}{\square_{x_0 \dots x_n} \text{sort}} \boxed{\vartheta}$ extends the signature by the declaration $\vartheta : [x_0 : A_0, \dots, x_n : A_n \vdash \text{sort}]$, and imposes the notational convention $\square_{x_0 \dots x_n} \triangleq \vartheta(x_0, \dots, x_n)$.

These notational conventions are permitted to omit arguments which are obvious from context, giving an informal counterpart to the notion of *implicit arguments* which appear in proof assistants like Agda [Nor09].

Operation declaration The operation formation rule $\frac{x_0 : A_0 \quad \dots \quad x_n : A_n}{\square_{x_0 \dots x_n} : B} \boxed{\vartheta}$ extends the signature by the declaration $\vartheta : [x_0 : A_0, \dots, x_n : A_n \vdash B]$, and imposes the notational convention $\square_{x_0 \dots x_n} \triangleq \vartheta(x_0 \dots x_n)$.

Sort axiom The sort equation rule $\frac{x_0 : A_0 \quad \dots \quad x_n : A_n}{A = B \text{ sort}} \boxed{=}$ extends the signature by the axiom $[x_0 : A_0, \dots, x_n : A_n \vdash A = B \text{ sort}]$.

Term axiom The term equation rule $\frac{x_0 : A_0 \quad \dots \quad x_n : A_n}{M = N : A} \boxed{=}$ extends the signature by the axiom $[x_0 : A_0, \dots, x_n : A_n \vdash M = N : A]$.

Example The theory of monoids can be written using our new notation as follows:

$$\frac{}{\text{ob sort}} \boxed{\text{ob}} \quad \frac{}{\text{id} : \text{ob}} \boxed{\text{id}} \quad \frac{x : \text{ob} \quad y : \text{ob}}{x \bullet y : \text{ob}} \boxed{\text{cmp}} \quad \frac{x : \text{ob}}{x \bullet \text{id} = x : \text{ob}} \boxed{=} \\ \frac{x : \text{ob}}{\text{id} \bullet x = x : \text{ob}} \boxed{=} \quad \frac{x : \text{ob} \quad y : \text{ob} \quad z : \text{ob}}{(x \bullet y) \bullet z = x \bullet (y \bullet z) : \text{ob}} \boxed{=}$$

1.5 Related work: logical frameworks

Generalized algebraic theories comprise one point in the space of *logical frameworks*, which are syntactic disciplines for formulating deductive systems. The purpose of a logical framework is to distinguish between the parts of a deductive system which are *particular* (for instance, the generators and equations) and the parts which are *universal* (for instance, the typing or binding discipline). Logical frameworks vary primarily in which aspects of deductive systems they treat as universal, negotiating the duality between *expressivity* and *utility*.

1.5.1 First-order algebraic theories

One of the most basic logical frameworks is that of *first-order algebra*, in which there are only atomic sorts, and contexts have the structure of a strictly associative cartesian product. The science of functorial semantics was developed first in the context of unsorted first-order algebraic theories by Lawvere [Law04] in 1968.

Like generalized algebraic theories, first-order algebraic theories lack any intrinsic binding structure. As such, they do not *natively* explain the universal syntactic phenomena which emanate from the lambda calculus and other languages with binding. In contrast, the dependent sorts of generalized algebraic theories lend themselves to a workable formalization of De Bruijn indices and explicit substitutions, which we have employed here.

1.5.2 Second-order algebraic theories

Fiore, Plotkin, and Turi [FPT99] initiated the scientific study of *second-order algebra* — which had already appeared in embryonic form as early as Aczel [Acz78] — a discipline which encompasses theories whose operations exhibit binding structure of one level, with variables that range over terms and metavariables which range over binders. The functorial semantics of second-order algebraic theories was developed by Fiore and Mahmoud [FM10]. A variation on second-order algebra which omits both equations and metavariables, but adds a novel notion of *indexed operation*, was employed by Harper [Har12b] to provide a syntactic discipline for formulating the syntax and semantics of programming languages.

1.5.3 Essentially algebraic theories

Essentially algebraic theories are the closest (semantic) relative to Cartmell’s generalized algebraic theories. Generalized algebraic theories provide dependently-sorted syntax for concepts which exhibit indexing; essentially algebraic theories capture these concepts using *fibration* rather than *parameterization*.

Every essentially algebraic theory can be presented as a generalized algebraic theory by axiomatizing proof-irrelevant predicates; generalized algebraic theories can likewise be transformed into essentially algebraic theories by taking the “total sorts” of families and using predicates to specify indices. When transforming an essentially algebraic theory into a generalized algebraic theory, one must choose which relations to express using parameterization and which to express using fibration. For this reason, Voevodsky observed that it is not correct to consider the two disciplines interchangeable [Voe13].

1.5.4 Martin-Löf’s Logical Framework

In Martin-Löf [Mar84], the use of “higher-level variables” (variables which range over binders) was introduced to the syntax of Intuitionistic Type Theory, a syntactic discipline called a *theory of expressions* in Nordström, Peterson, and Smith [NPS90]. This simply-typed higher-order logical framework was employed to systematize the treatment of variable binding in Intuitionistic Type Theory, which maintained at the time a separate *extrinsic* typing discipline which was defined on top of the simple arities.

The theory of expressions was an immediate precursor to dependently typed logical frameworks, principally Martin-Löf’s Logical Framework (LF) and the Edinburgh Logical Framework (ELF). Object theories formulated in the LF use the LF type structure to simultaneously express both their binding structure and their typing discipline: constants are added to a signature together with an LF-type, employing the ambient dependent

function type to achieve both binding (of higher level) and parameterization. Signatures in the LF can be extended with equations, analogous to the state of affairs in generalized algebraic theories.

Variable binding in LF Because object theories formulated in the LF inherit their binding discipline from the metalanguage, there is likewise no need to formalize contexts. Whereas in the generalized algebraic theory of categories with families (cwfs) which we recapitulate in [Section 3](#), we formalize the notion of a context, and then every operator takes as an argument its context Γ in addition to its other parameters, this part of the structure is implicit in the LF. A consequence is, however, that the LF must be revised or extended in order to support languages with exotic binding constructs and context effects, such as modalities.

1.5.5 Edinburgh Logical Framework

Introduced by Harper, Honsell, and Plotkin [HHP93], the Edinburgh Logical Framework (ELF) shares its type structure with Martin-Löf’s Logical Framework, but its notion of signature is different, and therefore its mode of use also differs. Whereas the LF allowed a signature to be extended with equations as is customary in algebra, the ELF was designed in order to enable a strict bijection (called *adequacy*) between ELF terms and object-theory derivations, *including* derivations of formal equality.

For this reason, while Martin-Löf’s Logical Framework is most suited to providing a syntactic discipline for object theories which encompasses both typing and formal equality, the Edinburgh Logical Framework is better adapted to situations in which one wishes to prove a syntactic metatheorem about a formal system by induction on its derivations. The Edinburgh Logical Framework, implemented in the Twelf proof assistant [PS99], has been used to formalize the syntactic metatheory of numerous logics and even programming languages [LCH07; HL07].

1.5.6 Perspective

The main axes of variation in logical frameworks are to be found in negotiating the universality of *type structure*, *binding structure* and *formal equality*. In LF and ELF, the question of binding structure is essentially subsumed by the type structure, which is treated as universal; in LF, formal equality is treated as universal, whereas in ELF it is treated as particular. Generalized algebraic theories represent a middle ground, in which dependently-sorted first-order syntax can be formalized up to formal equality, *including* the theory of De Bruijn indices and explicit substitutions.

While such an object-level formalization of binding structure can prove tedious, and necessarily results in a proliferation of axioms about how substitutions propagate through constructors, it is the most natural setting in which to develop an algebraic account of categories of models of type theory, such as categories with families or categories with attributes. The initial category with families (extended with further structure, such as dependent function types), then, serves as a suitable notion of type theory; indeed, the Logical Framework itself arises in this way.

The Logical Framework may be the most natural place to develop many object languages, but we have found generalized algebraic theories to be a useful intermediate point worth developing, if only to have a principled matrix in which to construct the next thousand logical frameworks.

1.6 Categories of signatures vs doctrines

In modern algebra, one considers two parallel perspectives on “notion of theory”:

1. There is the 1-categorical perspective, in which a notion of theory is given by (something equivalent to) a 1-category of signatures and interpretations. This is the perspective that we have followed in [Section 1.5](#), and indeed, in the rest of this note.
2. There is the 2-categorical perspective, in which a notion of theory is given by 2-category of theories (called a *doctrine*). This usually arises from a *universal characterization* of the basic structure (like finite products, finite limits, etc.), as opposed to a choice of basic structure.

For instance, the 2-categorical perspective on algebraic theories arises from the doctrine of categories with finite products and functors which preserve finite products. On the other hand, one can define the 1-category of first-order algebraic theories as either the category of algebraic signatures and interpretations \mathbf{Sign}_x , or the 1-category of categories equipped with a choice of strictly-associative finite products \mathbf{Law} . Both \mathbf{Sign}_x and \mathbf{Law} are equivalent; while \mathbf{Law} looks as though it might extend to a 2-categorical notion (because its objects are categories), it is fundamentally 1-categorical because the only sensible class of 2-cells would contain only identities [HP07].

Cartmell [Car78] treats generalized algebraic theories from a purely 1-categorical perspective. Theories \mathbb{T} are arranged into a 1-category \mathbf{GAT} , with morphisms $\mathbb{T} \rightarrow \mathbb{T}'$ given by interpretations of the language of \mathbb{T} into the language of \mathbb{T}' ; another 1-category of *contextual categories* \mathbf{Con} is defined which plays exactly the role of \mathbf{Law} in relation to \mathbf{GAT} , which is to forget the difference between derived and generating morphisms.

The 2-categorical perspective given by doctrines is essential for general semantics, but our immediate aim is more restricted: we are using semantic tools to prove theorems about syntax (such as canonicity, normalization, coherence, decidability of equality, etc.); toward these ends, the 1-categorical perspective is the simplest and most immediately adaptable, and we require only 1-categorical initiality results.

In contrast, to prove the equivalence of categories with families and locally cartesian closed categories, one must develop the 2-categorical notion of cwf rather than the 1-categorical notion that we develop here (i.e. the category of models of the theory of cwf); one does not obtain, for instance, a free locally cartesian closed category from the initial model of the theory of cwf [CCD17].

1.7 Algebraic semantics and initiality

Every generalized algebraic theory \mathbb{T} gives rise to a category of models $\mathbb{T}\text{-Alg}$; concretely, a model of \mathbb{T} is given by an interpretation of sorts and operations in families of sets.

A sort “ $\vdash_{\mathbb{T}} A \text{ sort}$ ” is interpreted as a set $\llbracket A \rrbracket$, whereas a sort “ $x : A \vdash_{\mathbb{T}} B \text{ sort}$ ” is interpreted as a $\llbracket A \rrbracket$ -indexed family of sets $\llbracket B \rrbracket$, and so on.

Every interpretation $I : \mathbb{T} \rightarrow \mathbb{T}'$ in **GAT** induces a restriction of algebras $I^* : \mathbb{T}'\text{-Alg} \rightarrow \mathbb{T}\text{-Alg}$ (precomposition with the interpretation), and this has a left adjoint $I_!$. Considering the universal interpretation $!_{\mathbb{T}} : \emptyset \rightarrow \mathbb{T}$, we observe that the codomain $\emptyset\text{-Alg}$ of its restriction functor is actually the terminal category: there is only one model of the theory with no sorts and no operations; from the left adjoint, we therefore obtain an initial object in $\mathbb{T}\text{-Alg}$, i.e. an initial model.

The construction of the left adjoint involves adjoining new rules to the *term model* (Lindenbaum-Tarski model) of the given theory; Cartmell [Car78] constructs this term model in painstaking detail, but merely observes without proof that the left adjoint to the evident restriction functor exists.

The existence of these left adjoints is proved in more detail in the context of *essentially algebraic theories* by Palmgren and Vickers [PV07]. We are not aware of a similarly detailed proof for generalized algebraic theories in the literature. If one is unsatisfied with this state of affairs, one can observe that every generalized algebraic theory induces an essentially algebraic theory with an equivalent category of models, and then transport the initial object along this equivalence.

2 Warming up: the theory of categories

We define the generalized algebraic theory of categories, $\mathbb{T}_{\text{cat}} \text{thy}$.

$$\begin{array}{c}
\frac{}{\text{sort}} \boxed{\text{ob}} \qquad \frac{\Delta : \text{ob} \quad \Gamma : \text{ob}}{\Delta \Rightarrow \Gamma \text{ sort}} \boxed{\text{hom}} \qquad \frac{\Gamma : \text{ob}}{\text{id} : \Gamma \Rightarrow \Gamma} \boxed{\text{id}} \\
\\
\frac{H, \Delta, \Gamma : \text{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad \delta : H \Rightarrow \Delta}{\gamma \circ \delta : H \Rightarrow \Gamma} \boxed{\text{cmp}} \qquad \frac{\Delta, \Gamma : \text{ob} \quad \gamma : \Delta \Rightarrow \Gamma}{\gamma \circ \text{id} = \gamma : \Delta \Rightarrow \Gamma} \boxed{=} \\
\\
\frac{\Delta, \Gamma : \text{ob} \quad \gamma : \Delta \Rightarrow \Gamma}{\text{id} \circ \gamma = \gamma : \Delta \Rightarrow \Gamma} \boxed{=} \\
\\
\frac{\Xi, H, \Delta, \Gamma : \text{ob} \quad \eta : \Xi \Rightarrow H \quad \delta : H \Rightarrow \Delta \quad \gamma : \Delta \Rightarrow \Gamma}{(\gamma \circ \delta) \circ \eta = \gamma \circ (\delta \circ \eta) : \Xi \Rightarrow \Gamma} \boxed{=}
\end{array}$$

The collection of models of \mathbb{T}_{cat} induces a *1-categorical* notion of “category”, which we will exploit in our formulation of algebraic cwf’s.

3 Algebraic cwf’s as a notion of type theory

In semantics, categories with families (cwf’s) are a familiar doctrine for type theories [Dyb96; Fio12], naturally organized into a 2-category; but another perspective is given by the *generalized algebraic theory of cwf’s*, whose 1-category of models and homomorphisms gives a more algebraic and strict notion of type theory, in which all

structure is chosen globally and homomorphisms are arranged to preserve it on the nose. We define the theory of cwfs, $\mathbb{T}_{\text{cwf}} \text{thy}$, by extending \mathbb{T}_{cat} with the following sorts, operations and axioms.

Types and elements

$$\frac{\Gamma : \mathbf{ob}}{\mathbf{T}\mathbf{y}(\Gamma) \text{ sort}} \boxed{\mathbf{T}\mathbf{y}} \qquad \frac{\Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{T}\mathbf{y}(\Gamma)}{A[\gamma] : \mathbf{T}\mathbf{y}(\Delta)} \boxed{\mathbf{T}\mathbf{y}/\mathbf{act}}$$

$$\frac{\Gamma : \mathbf{ob} \quad A : \mathbf{T}\mathbf{y}(\Gamma)}{A[\mathbf{id}] = A : \mathbf{T}\mathbf{y}(\Gamma)} \boxed{=} \qquad \frac{H, \Delta, \Gamma : \mathbf{ob} \quad \delta : H \Rightarrow \Delta \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{T}\mathbf{y}(\Gamma)}{A[\gamma \circ \delta] = A[\gamma][\delta] : \mathbf{T}\mathbf{y}(H)} \boxed{=}$$

$$\frac{\Gamma : \mathbf{ob} \quad A : \mathbf{T}\mathbf{y}(\Gamma)}{\mathbf{E}\mathbf{l}(\Gamma \vdash A) \text{ sort}} \boxed{\mathbf{E}\mathbf{l}}$$

$$\frac{\Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{T}\mathbf{y}(\Gamma) \quad M : \mathbf{E}\mathbf{l}(\Gamma \vdash A)}{M[\gamma] : \mathbf{E}\mathbf{l}(\Delta \vdash A[\gamma])} \boxed{\mathbf{E}\mathbf{l}/\mathbf{act}}$$

$$\frac{\Gamma : \mathbf{ob} \quad A : \mathbf{T}\mathbf{y}(\Gamma) \quad M : \mathbf{E}\mathbf{l}(\Gamma \vdash A)}{M[\mathbf{id}] = M : \mathbf{E}\mathbf{l}(\Gamma \vdash M)} \boxed{=}$$

$$\frac{H, \Delta, \Gamma : \mathbf{ob} \quad \delta : H \Rightarrow \Delta \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{T}\mathbf{y}(\Gamma) \quad a : \mathbf{E}\mathbf{l}(\Gamma \vdash A)}{a[\gamma \circ \delta] = a[\gamma][\delta] : \mathbf{E}\mathbf{l}(H \vdash A[\gamma \circ \delta])} \boxed{=}$$

Terminal context

$$\frac{}{\cdot : \mathbf{ob}} \boxed{\mathbf{emp}} \qquad \frac{\Gamma : \mathbf{ob}}{! : \Gamma \Rightarrow \cdot} \boxed{!} \qquad \frac{\Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma}{! \circ \gamma = ! : \Gamma \Rightarrow \cdot} \boxed{=} \qquad \frac{}{\mathbf{id} = ! : \cdot \Rightarrow \cdot} \boxed{=}$$

Context comprehension

$$\frac{\Gamma : \mathbf{ob} \quad A : \mathbf{T}\mathbf{y}(\Gamma)}{\Gamma.A : \mathbf{ob}} \boxed{\mathbf{ext}}$$

$$\frac{\Delta, \Gamma : \mathbf{ob} \quad A : \mathbf{T}\mathbf{y}(\Gamma) \quad \gamma : \Delta \Rightarrow \Gamma \quad N : \mathbf{E}\mathbf{l}(\Delta \vdash A[\gamma])}{\langle \gamma, N \rangle : \Delta \Rightarrow \Gamma.A} \boxed{\mathbf{snoc}}$$

$$\frac{\Gamma : \mathbf{ob} \quad A : \mathbf{T}\mathbf{y}(\Gamma)}{\mathbf{p} : \Gamma.A \Rightarrow \Gamma} \boxed{\mathbf{p}} \qquad \frac{\Gamma : \mathbf{ob} \quad A : \mathbf{T}\mathbf{y}(\Gamma)}{\mathbf{q} : \mathbf{E}\mathbf{l}(\Gamma.A \vdash A[\mathbf{p}])} \boxed{\mathbf{q}}$$

$$\frac{\Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{T}\mathbf{y}(\Gamma) \quad M : \mathbf{E}\mathbf{l}(\Delta \vdash A[\gamma])}{\mathbf{p} \circ \langle \gamma, M \rangle = \gamma : \Delta \Rightarrow \Gamma} \boxed{=}$$

$$\frac{\Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{Ty}(\Gamma) \quad M : \mathbf{El}(\Delta \vdash A[\gamma])}{\mathbf{q}[\langle \gamma, M \rangle] = M : \mathbf{El}(\Delta \vdash A[\gamma])} \quad \square$$

$$\frac{H, \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad \delta : H \Rightarrow \Delta \quad A : \mathbf{Ty}(\Gamma) \quad M : \mathbf{El}(\Delta \vdash A[\gamma])}{\langle \gamma, M \rangle \circ \delta = \langle \gamma \circ \delta, M[\delta] \rangle : H \Rightarrow \Gamma.A} \quad \square$$

$$\frac{\Gamma : \mathbf{ob} \quad A : \mathbf{Ty}(\Gamma)}{\mathbf{id} = \langle \mathbf{p}, \mathbf{q} \rangle : \Gamma.A \Rightarrow \Gamma.A} \quad \square$$

The highly general results of Cartmell give rise to a category of models of \mathbb{T}_{cwf} , which has an initial object, inducing a functorial semantics in the sense of Lawvere [Law04]. This initial object is the *free Martin-Löf Type Theory without any connectives*, and with a bit of labor, it can be seen to be isomorphic to the Lindenbaum-Tarski model generated by the raw syntax of pure MLTT without connectives.

By extending the theory \mathbb{T}_{cwf} with further structure, such as dependent function types, dependent pair types, universes, identity types, cubical interval, etc., nearly every conceivable extension of MLTT can be obtained, together with the appropriate category of models and homomorphisms. In this note, part of our intention is to show how to obtain some of these extensions which are commonly believed to fall outside the range of applicability of algebraic techniques.

We emphasize that establishing the equivalence between these initial models and Lindenbaum-Tarski models obtained by constraining and then quotienting the “raw syntax” is laborious in a technical sense, and may in the future be seen to be superfluous: while some have fetishized the raw syntax of type theory to such a degree that the specific textual/linear rendering of name binding (and the attendant α -convention) has been elevated from an expedient notation to an actual object of study, we predict that the (abstractly presented) initial models for algebraic type theory will ultimately be taken as definitive in the study of type-theoretic syntax, as advocated in Castellan, Clairambault, and Dybjer [CCD17].²

Traditional presentations of type theory using raw syntax and unstructured masses of inference rules took hold in the years before a workable account of dependently typed universal algebra had been obtained; rather than making a virtue out of ancient necessity, we hold that the most suitable notion of syntax arises abstractly in a purely algebraic way (much like how previous generations of type theorists had already eschewed the antique identification of syntax with punctuated sequences of symbols [Acz78]).

From the perspective of a *user* of type theory, we stress, there is no serious gap presented by the abstract syntax induced by the initial cwf; in fact, concrete computerized implementations of type theory tend to be much closer to the abstract syntax of the initial cwf than to the “raw” syntax which some have insisted is a primary object of study.

²Castellan, Clairambault, and Dybjer argue that it is circular to obtain the initial cwf from the generalized algebraic theory of cwfs, because “the notion of a generalised algebraic theory is itself based on dependent type theory”. On the contrary, the notion of generalized algebraic theory is developed primitively by Cartmell [Car78; Car86] in the ambient set theory without making use of any pre-existing type-theoretic machinery. Therefore, we maintain that no circularity ensues from the algebraic generation of the initial cwfs; *a posteriori*, the latent cwf structure involved in defining the notion of generalized algebraic theories can be observed, underscoring the unity between the metatheory of type theory and type theory itself.

4 Algebraic type hierarchies

We begin by showing how to extend the theory of cwfs to include predicative hierarchies of type systems, giving an algebraic treatment to Coquand's *cumulative cwfs* [Coq18]; then we will show how to add a hierarchy of universes à la Russell in a modular way. We will replace the **Ty** operator with something parameterized in a universe level α , giving the sort of types of level α .

Theory of type levels

$$\begin{array}{c}
 \frac{}{\mathbf{lvl} \text{ sort}} \boxed{\mathbf{lvl}} \quad \frac{}{0 : \mathbf{lvl}} \boxed{\mathbf{z}} \quad \frac{\alpha : \mathbf{lvl}}{\alpha + 1 : \mathbf{lvl}} \boxed{\mathbf{s}} \quad \frac{\alpha, \beta : \mathbf{lvl}}{\alpha < \beta \text{ sort}} \boxed{\mathbf{lt}} \quad \frac{\alpha : \mathbf{lvl}}{0 < \alpha + 1} \boxed{\mathbf{lt/z}} \\
 \\
 \frac{\alpha : \mathbf{lvl}}{\alpha < \alpha + 1} \boxed{\mathbf{lt/s}} \quad \frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta}{\alpha + 1 < \beta + 1} \boxed{\mathbf{lt/lift}} \\
 \\
 \frac{\alpha, \alpha', \beta : \mathbf{lvl} \quad p : \alpha < \alpha' \quad q : \alpha' < \beta}{\alpha < \beta} \boxed{\mathbf{lt/cmp}} \quad \frac{\alpha, \beta : \mathbf{lvl} \quad p, q : \alpha < \beta}{p = q : \alpha < \beta} \boxed{=}
 \end{array}$$

Types and elements

$$\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob}}{\mathbf{Ty}_\alpha(\Gamma) \text{ sort}} \boxed{\mathbf{Ty}} \quad \frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma)}{\mathbf{El}(\Gamma \vdash A) \text{ sort}} \boxed{\mathbf{El}} \quad \dots$$

We will not recapitulate the remainder of the theory (e.g. context comprehension), noting that it proceeds by adding $\alpha : \mathbf{lvl}$ to most telescopes. Instead, we focus on what must be added to achieve the *algebraic* version of cumulativity for types, and then (algebraically) cumulative universes à la Russell.

Remark 4.1 (Algebraic cumulativity). *In this note, we consider an algebraic form of cumulativity which does not require any kind of subtyping. Instead, we have explicit shifts between universes which are ensured by algebraic laws to commute with all the connectives of type theory: to put it crudely, we require $\uparrow A \times B = \uparrow A \times \uparrow B$.*

4.1 Algebraic cumulativity and lifting

In order to achieve cumulativity, we add an operator which lifts a type of level α to level $\beta > \alpha$:

$$\begin{array}{c}
 \text{LIFT FORMATION} \\
 \frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma)}{\uparrow_\alpha^\beta A : \mathbf{Ty}_\beta(\Gamma)} \boxed{\mathbf{lift}} \\
 \\
 \text{LIFT SUBSTITUTION} \\
 \frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{Ty}_\alpha(\Gamma)}{(\uparrow_\alpha^\beta A)[\gamma] = \uparrow_\alpha^\beta(A[\gamma]) : \mathbf{Ty}_\alpha(\Delta)} \boxed{=}
 \end{array}$$

LIFT COMPOSITION

$$\frac{\alpha, \alpha', \beta : \mathbf{lvl} \quad p : \alpha < \alpha' \quad q : \alpha' < \beta \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma)}{\uparrow_{\alpha'}^\beta \uparrow_\alpha^{\alpha'} A = \uparrow_\alpha^\beta A : \mathbf{Ty}_\beta(\Gamma)} \quad \boxed{=}$$

ELEMENT LIFTING

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma)}{\mathbf{El}(\Gamma \vdash A) = \mathbf{El}(\Gamma \vdash \uparrow_\alpha^\beta A) \text{ sort}} \quad \boxed{=}$$

CONTEXT LIFTING

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma)}{\Gamma.A = \Gamma. \uparrow_\alpha^\beta A : \mathbf{ob}} \quad \boxed{=}$$

4.2 Type-theoretic connectives

Adding connectives (like dependent function types) to the algebraic theory of cumulative cwfs is simple, but we must take care to ensure that level shifting commutes through the connectives properly. In the case of dependent function types, we go beyond the usual only in adding enough axioms to make the shifts “irrelevant” (for instance, equating $\uparrow_\alpha^\beta \Pi(A, B)$ and $\Pi(\uparrow_\alpha^\beta A, \uparrow_\alpha^\beta B)$).

PI FORMATION

$$\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A)}{\Pi(A, B) : \mathbf{Ty}_\alpha(\Gamma)} \quad \boxed{\Pi}$$

PI LIFTING

$$\frac{\alpha, \beta : \mathbf{lvl} \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A)}{\uparrow_\alpha^\beta \Pi(A, B) = \Pi(\uparrow_\alpha^\beta A, \uparrow_\alpha^\beta B) : \mathbf{Ty}_\beta(\Gamma)} \quad \boxed{=}$$

PI SUBSTITUTION

$$\frac{\alpha : \mathbf{lvl} \quad \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A)}{\Pi(A, B)[\gamma] = \Pi(A[\gamma], B[\langle \gamma \circ \mathbf{p}, \mathbf{q} \rangle]) : \mathbf{Ty}_\alpha(\Delta)} \quad \boxed{=}$$

PI INTRODUCTION

$$\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A) \quad M : \mathbf{El}(\Gamma.A \vdash B)}{\lambda(M) : \mathbf{El}(\Gamma \vdash \Pi(A, B))} \quad \boxed{\lambda}$$

LAMBDA LIFTING NATURALITY

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A) \quad M : \mathbf{El}(\Gamma.A \vdash B)}{\lambda\langle \beta, \Gamma, \uparrow_\alpha^\beta A, \uparrow_\alpha^\beta B, M \rangle = \lambda\langle \alpha, \Gamma, A, B, M \rangle : \mathbf{El}(\Gamma \vdash \Pi(A, B))} \quad \boxed{=}$$

LAMBDA SUBSTITUTION

$$\frac{\alpha : \mathbf{lvl} \quad \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A) \quad M : \mathbf{El}(\Gamma.A \vdash B)}{\lambda(M)[\gamma] = \lambda(M[\langle \gamma \circ \mathbf{p}, \mathbf{q} \rangle]) : \mathbf{El}(\Delta \vdash \Pi(A[\gamma], B[\langle \gamma \circ \mathbf{p}, \mathbf{q} \rangle]))} \quad \boxed{=}$$

PI ELIMINATION

$$\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A) \quad M : \mathbf{El}(\Gamma \vdash \Pi(A, B)) \quad N : \mathbf{El}(\Gamma \vdash A)}{\mathbf{app}(M, N) : \mathbf{El}(\Gamma \vdash B[\langle \mathbf{id}, N \rangle])} \quad \boxed{\mathbf{app}}$$

APP LIFTING NATURALITY

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A) \quad M : \mathbf{El}(\Gamma \vdash \Pi(A, B)) \quad N : \mathbf{El}(\Gamma \vdash A)}{\mathbf{app}\langle \beta, \Gamma, \uparrow_\alpha^\beta A, \uparrow_\alpha^\beta B, M, N \rangle = \mathbf{app}\langle \alpha, \Gamma, A, B, M, N \rangle : \mathbf{El}(\Gamma \vdash B[\langle \mathbf{id}, N \rangle])} \quad \square$$

APP SUBSTITUTION

$$\frac{\alpha : \mathbf{lvl} \quad \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A) \quad M : \mathbf{El}(\Gamma \vdash \Pi(A, B)) \quad N : \mathbf{El}(\Gamma \vdash A)}{\mathbf{app}(M, N)[\gamma] = \mathbf{app}(M[\gamma], N[\gamma]) : \mathbf{El}(\Delta \vdash B[\langle \gamma, N[\gamma] \rangle])} \quad \square$$

PI UNICITY

$$\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A) \quad M : \mathbf{El}(\Gamma \vdash \Pi(A, B))}{M = \lambda(\mathbf{app}(M[\mathbf{p}], \mathbf{q})) : \mathbf{El}(\Gamma \vdash \Pi(A, B))} \quad \square$$

PI COMPUTATION

$$\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob} \quad A : \mathbf{Ty}_\alpha(\Gamma) \quad B : \mathbf{Ty}_\alpha(\Gamma.A) \quad M : \mathbf{El}(\Gamma.A \vdash B) \quad N : \mathbf{El}(\Gamma \vdash A)}{\mathbf{app}(M, N) = M[\langle \mathbf{id}, N \rangle] : \mathbf{El}(\Gamma \vdash B[\langle \mathbf{id}, N \rangle])} \quad \square$$

4.3 Universes à la Russell

Now we see how to add universes à la Russell to the theory of cumulative cwfs. First we add generators for the universe types themselves.

UNIVERSE FORMATION

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Gamma : \mathbf{ob}}{\mathbf{U}_\alpha : \mathbf{Ty}_\beta(\Gamma)} \quad \square$$

UNIVERSE LIFTING

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \alpha' \quad q : \alpha < \beta \quad \Gamma : \mathbf{ob}}{\uparrow_{\alpha'}^\beta \mathbf{U}_\alpha = \mathbf{U}_\alpha : \mathbf{Ty}_\beta(\Gamma)} \quad \square$$

UNIVERSE SUBSTITUTION

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma}{\mathbf{U}_\alpha[\gamma] = \mathbf{U}_\alpha : \mathbf{Ty}_\beta(\Delta)} \quad \square$$

To characterize the elements of the universes, it suffices to impose a sort equation between the $\mathbf{El}(\Gamma \vdash \mathbf{U}_\alpha)$ and $\mathbf{Ty}_\alpha(\Gamma)$:

UNIVERSE ELEMENTS

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Gamma : \mathbf{ob}}{\mathbf{El}(\Gamma \vdash \mathbf{U}\langle \alpha, \beta, p, \Gamma \rangle) = \mathbf{Ty}_\alpha(\Gamma) \text{ sort}} \quad \square$$

4.4 A base type and constants

We extend our theory with a base type and two constants; this will be useful for illustrating a non-trivial metatheorem using algebraic methods.

FORMATION

$$\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob}}{\mathbf{color} : \mathbf{Ty}_\alpha(\Gamma)} \quad \square$$

OBS LIFTING

$$\frac{\alpha, \beta : \mathbf{lvl} \quad p : \alpha < \beta \quad \Gamma : \mathbf{ob}}{\uparrow_\alpha^\beta \mathbf{color} = \mathbf{color} : \mathbf{Ty}_\beta(\Gamma)} \quad \square$$

$$\begin{array}{c}
\text{OBS SUBSTITUTION} \\
\frac{\alpha : \mathbf{lvl} \quad \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma}{\mathbf{color}[\gamma] = \mathbf{color} : \mathbf{Ty}_\alpha(\Delta)} \quad \boxed{=}
\end{array}
\qquad
\begin{array}{c}
\text{INTRODUCTION 1} \\
\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob}}{\mathbf{red} : \mathbf{El}(\Gamma \vdash \mathbf{color})} \quad \boxed{\mathbf{red}}
\end{array}$$

$$\begin{array}{c}
\text{INTRODUCTION 2} \\
\frac{\alpha : \mathbf{lvl} \quad \Gamma : \mathbf{ob}}{\mathbf{green} : \mathbf{El}(\Gamma \vdash \mathbf{color})} \quad \boxed{\mathbf{green}}
\end{array}
\qquad
\begin{array}{c}
\text{SUBSTITUTION 1} \\
\frac{\alpha : \mathbf{lvl} \quad \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma}{\mathbf{red}[\gamma] = \mathbf{red} : \mathbf{El}(\Delta \vdash \mathbf{color})} \quad \boxed{=}
\end{array}$$

$$\begin{array}{c}
\text{SUBSTITUTION 2} \\
\frac{\alpha : \mathbf{lvl} \quad \Delta, \Gamma : \mathbf{ob} \quad \gamma : \Delta \Rightarrow \Gamma}{\mathbf{green}[\gamma] = \mathbf{green} : \mathbf{El}(\Delta \vdash \mathbf{color})} \quad \boxed{=}
\end{array}$$

We do not add any elimination rules for the **color** type: its role is only to serve as an *observable* with which to phrase a canonicity result.

5 Canonicity for Martin-Löf Type Theory

One of the major benefits of the algebraic approach to defining type theories is that powerful *semantic tools* are immediately available for developing syntactic metatheory. As an example, we prove canonicity for Martin-Löf Type Theory with dependent function types and a hierarchy of universes à la Russell, following Coquand [Coq18], Shulman [Shu15] and Martin-Löf [Mar75a]. We will write $\mathbb{T}_{\text{cwf}}^\Pi$ for the theory defined in the previous sections.

5.1 The computability construction

Starting from any model $\mathcal{C} : \mathbb{T}_{\text{cwf}}^\Pi\text{-Alg}$, we will show how to construct a *new* model $\bar{\mathcal{C}} : \mathbb{T}_{\text{cwf}}^\Pi\text{-Alg}$ which *glues* each context from \mathcal{C} together with a *logical family* over its closed elements, following Coquand [Coq18].³ The logical families construction is a modern and thoroughly constructive version of the method of computability, in which predicates and relations are eschewed in favor of proof-relevant families.

A consequence of this streamlined approach is that there is no need to consider raw terms and partial equivalence relations, something which had previously been essential for developing the syntactic metatheory of dependent type theory with universes.

Remark 5.1 (Categorical digression). *While it is not our intention here to explain categorical gluing for dependent type theory (see Coquand [Coq18] and Shulman [Shu15]), we briefly observe that what we will unleash in type-theoretic language below can be understood more abstractly as an instance of the gluing construction, in which the fundamental*

³We prefer the term “logical family” to the more common term “proof-relevant logical predicate”.

fibration is pulled back along the global sections functor of the category of contexts:

$$\begin{array}{ccc}
\bar{\mathcal{C}} & \longrightarrow & \mathbf{Set}^{\rightarrow} \\
\downarrow & \lrcorner & \downarrow \pi_{\text{cod}} \\
\mathcal{C} & \xrightarrow{\mathcal{C}(\cdot, -)} & \mathbf{Set}
\end{array}$$

The remainder of the work, then, is to observe that the cwf structure lifts from \mathcal{C} to $\bar{\mathcal{C}}$. All aspects of the interpretation are forced except for the interpretation of the base type and the universe, which we are free to choose; in fact, it is the choice of \mathcal{C} -interpretation of the base type in which the essence of the proof of canonicity lies.

Assumption 5.2 (Set-theoretic universes). As a simplifying move, we assume a transfinite hierarchy of Grothendieck universes \mathcal{V}_α in the ambient set theory for $\alpha \in \{0, 1, \dots, \omega\}$. The universe \mathcal{V}_ω is only a convenience, and could be eliminated using a more verbose schematic or fibered construction.

To exhibit $\bar{\mathcal{C}}$ as a model, we must determine families of sets $[[\vartheta]]_{\bar{\mathcal{C}}}$ to interpret each sort symbol $\vartheta : [\Psi \vdash \text{sort}] \in \mathbb{T}_{\text{cwf}}^{\Pi}$; and likewise for operation symbols. We begin by imposing some notation:

1. We will write $|\Gamma|$ for $[[\mathbf{hom}]]_{\mathcal{C}}\langle[[\cdot]]_{\mathcal{C}}, \Gamma\rangle$ when $\Gamma \in [[\mathbf{ob}]]_{\mathcal{C}}$.
2. We will write $|A|$ for $[[\mathbf{E1}]]_{\mathcal{C}}\langle\alpha, [[\cdot]]_{\mathcal{C}}, A\rangle$ when $A \in [[\mathbf{Ty}]]_{\mathcal{C}}\langle\alpha, [[\cdot]]_{\mathcal{C}}\rangle$.
3. When it is unambiguous, we will use the notation of the algebraic theory itself to refer to objects in the models \mathcal{C} and $\bar{\mathcal{C}}$; for instance, $\Delta \Rightarrow \Gamma$ rather than $[[\mathbf{hom}]]_{\mathcal{C}}\langle\Delta, \Gamma\rangle$, and $\bar{\Delta} \Rightarrow \bar{\Gamma}$ rather than $[[\mathbf{hom}]]_{\bar{\mathcal{C}}}\langle\bar{\Delta}, \bar{\Gamma}\rangle$. To avoid ambiguity, we will write the $\bar{\mathcal{C}}$ -interpretations of unary sorts and operations with an overline, e.g. $\overline{\mathbf{ob}}$ for $[[\mathbf{ob}]]_{\bar{\mathcal{C}}}$.

As a notational convenience, we will generally write $\bar{X} \equiv (X, X^{\bullet})$ for something in the computability model, the gluing of X from \mathcal{C} with its “realizer” X^{\bullet} .

Contexts and substitutions We are now ready to define the glued interpretation of the contexts and substitutions.

$$\overline{\mathbf{ob}} = \coprod_{\Gamma \in \mathbf{ob}} (|\Gamma| \rightarrow \mathcal{V}_\omega) \quad (\text{objects})$$

We will write $\bar{\Gamma}$ for $(\Gamma, \Gamma^{\bullet}) \in \overline{\mathbf{ob}}$. Substitutions are interpreted as \mathcal{C} -substitutions together with realizers of the logical family.

$$\bar{\Delta} \Rightarrow \bar{\Gamma} = \coprod_{\gamma \in \Delta \Rightarrow \Gamma} \prod_{\delta \in |\Delta|} \prod_{\delta^{\bullet} \in \Delta^{\bullet} \delta} \Gamma^{\bullet}(\gamma \circ \delta) \quad (\text{substitutions})$$

When interpreting operations $\bar{\vartheta}$ whose sort-interpretations in $\bar{\mathcal{C}}$ glue a \mathcal{C} -construct together with a realizer, we follow convention of merely exhibiting the realizer ϑ^{\bullet} rather than $(\vartheta, \vartheta^{\bullet})$.

$$\begin{aligned}
\mathbf{id}^{\bullet} \gamma \gamma^{\bullet} &= \gamma^{\bullet} && (\text{identity substitution}) \\
(\bar{\gamma} \circ \bar{\delta})^{\bullet} \eta \eta^{\bullet} &= \gamma^{\bullet}(\delta \circ \eta)(\delta^{\bullet} \eta \eta^{\bullet}) && (\text{substitution composition})
\end{aligned}$$

Levels First, observe that we can embed any natural number α as a term of sort **lvl** in $\mathbb{T}_{\text{cwf}}^{\Pi}$; write $[\alpha]$ for its interpretation in \mathcal{C} . Then, we interpret **lvl** in the computability model as a \mathcal{C} -level together with a compatible natural number (its “realizer”):

$$\begin{aligned}\overline{\text{lvl}} &= \coprod_{\alpha \in \text{lvl}} \{\alpha^{\bullet} \in \mathbb{N} \mid [\alpha^{\bullet}] = \alpha\} \\ \bar{z} &= (z, 0) \\ \mathbf{s}(\bar{\alpha}) &= (\mathbf{s}(\alpha), \alpha^{\bullet} + 1)\end{aligned}$$

Writing $\bar{\alpha}$ for $(\alpha, \alpha^{\bullet}) \in \overline{\text{lvl}}$, we interpret the order on levels in the obvious way:

$$(\bar{\alpha} < \bar{\beta}) = \{p : \alpha < \beta \mid \alpha^{\bullet} < \beta^{\bullet}\}$$

The interpretation is always subsingleton, so it validates the proof irrelevance axiom that we imposed on **lvl**.

Types and elements A type is interpreted as a \mathcal{C} -type together with a logical family of the appropriate size defined over its closed instances:

$$\begin{aligned}\mathbf{T}_{\bar{\alpha}}(\bar{\Gamma}) &= \coprod_{A \in \mathbf{T}_{\gamma_{\alpha}(\Gamma)}} \prod_{\gamma \in |\Gamma|} \prod_{\gamma^{\bullet} \in \Gamma^{\bullet} \gamma} (|A[\gamma]| \rightarrow \mathcal{V}_{\alpha^{\bullet}}) && \text{(types)} \\ \mathbf{El}(\bar{\Gamma} \vdash \bar{A}) &= \coprod_{M \in \mathbf{El}(\Gamma \vdash A)} \prod_{\gamma \in |\Gamma|} \prod_{\gamma^{\bullet} \in \Gamma^{\bullet} \gamma} A^{\bullet} \gamma \gamma^{\bullet} M[\gamma] && \text{(elements)}\end{aligned}$$

Terminal context

$$\begin{aligned}\bullet \eta &= \{\star\} && \text{(terminal context)} \\ ! \bullet \eta \eta^{\bullet} &= \star && \text{(universal substitution)}\end{aligned}$$

Context comprehension

$$\begin{aligned}\bar{\Gamma}. \bar{A} \bullet \eta &= \coprod_{\gamma^{\bullet} \in \Gamma^{\bullet}(\mathbf{p} \circ \eta)} A^{\bullet}(\mathbf{p} \circ \eta) \gamma^{\bullet}(\mathbf{q}[\eta]) && \text{(context extension)} \\ \langle \bar{\gamma}, \bar{N} \rangle \bullet \delta \delta^{\bullet} &= (\gamma^{\bullet} \delta \delta^{\bullet}, N^{\bullet} \delta \delta^{\bullet}) && \text{(substitution extension)} \\ \bar{\mathbf{p}} \eta(\gamma^{\bullet}, N^{\bullet}) &= \gamma^{\bullet} && \text{(projection)} \\ \bar{\mathbf{q}} \eta(\gamma^{\bullet}, N^{\bullet}) &= N^{\bullet} && \text{(variables)}\end{aligned}$$

Type lifting The following interpretation of type lifting is well-defined, because we have imposed the axiom that the lifting of a type has the same elements as the type itself, and because the set-theoretic universe hierarchy \mathcal{V}_{α} is cumulative.

$$\left(\uparrow_{\bar{\alpha}}^{\bar{\beta}} \bar{A} \right)^{\bullet} \gamma \gamma^{\bullet} M = A^{\bullet} \gamma \gamma^{\bullet} M \quad \text{(type lifting)}$$

Dependent function types

$$\begin{aligned}\Pi(\bar{A}, \bar{B}) \bullet \gamma \gamma^{\bullet} M &= \prod_{N \in |A[\gamma]|} \prod_{N^{\bullet} \in A^{\bullet} \gamma \gamma^{\bullet} N} B^{\bullet} \langle \gamma, N \rangle (\gamma^{\bullet}, N^{\bullet}) (\mathbf{app}(M, N)) && \text{(formation)} \\ \lambda(\bar{M}) \bullet \gamma \gamma^{\bullet} N N^{\bullet} &= M^{\bullet} \langle \gamma, N \rangle (\gamma^{\bullet}, N^{\bullet}) && \text{(introduction)} \\ \mathbf{app}(\bar{M}, \bar{N}) \bullet \gamma \gamma^{\bullet} &= M^{\bullet} \gamma \gamma^{\bullet} N N^{\bullet} && \text{(elimination)}\end{aligned}$$

Universes à la Russell A realizer for an element of the $\bar{\alpha}$ th universe is an α^\bullet -small logical family over its closed elements:

$$\mathbf{U}_{\bar{\alpha}}^\bullet \gamma \gamma^\bullet A = |A| \rightarrow \mathcal{V}_{\alpha^\bullet} \quad (\text{formation})$$

Fixing $\bar{\alpha}, \bar{\beta} \in \mathbf{Ivl}$ with $\bar{p} : \bar{\alpha} < \bar{\beta}$ and $\bar{\Gamma} \in \mathbf{ob}$, we need to see that $\mathbf{El}(\bar{\Gamma} \vdash \mathbf{U}_{\bar{\alpha}}) = \mathbf{Ty}_{\bar{\alpha}}(\bar{\Gamma})$. Calculate:

$$\begin{aligned} \mathbf{El}(\bar{\Gamma} \vdash \mathbf{U}_{\bar{\alpha}}) &= \coprod_{A \in \mathbf{El}(\bar{\Gamma} \vdash \mathbf{U}_{\bar{\alpha}})} \prod_{\gamma \in |\Gamma|} \prod_{\gamma^\bullet \in \Gamma^\bullet \gamma} \mathbf{U}_{\bar{\alpha}}^\bullet \gamma \gamma^\bullet A[\gamma] && (\text{by def.}) \\ &= \coprod_{A \in \mathbf{Ty}_{\bar{\alpha}}(\bar{\Gamma})} \prod_{\gamma \in |\Gamma|} \prod_{\gamma^\bullet \in \Gamma^\bullet \gamma} \mathbf{U}_{\bar{\alpha}}^\bullet \gamma \gamma^\bullet A[\gamma] && (\mathcal{C} \text{ has universes}) \\ &= \coprod_{A \in \mathbf{Ty}_{\bar{\alpha}}(\bar{\Gamma})} \prod_{\gamma \in |\Gamma|} \prod_{\gamma^\bullet \in \Gamma^\bullet \gamma} (|A[\gamma]| \rightarrow \mathcal{V}_{\alpha^\bullet}) && (\text{by def.}) \\ &= \mathbf{Ty}_{\bar{\alpha}}(\bar{\Gamma}) && (\text{by def.}) \end{aligned}$$

Base type Finally, we give the interpretation of the base type.

$$\begin{aligned} \mathbf{color}^\bullet \gamma \gamma^\bullet M &= \{\star \mid M = \mathbf{red}\} + \{\star \mid M = \mathbf{green}\} \\ \mathbf{red}^\bullet \gamma \gamma^\bullet &= \mathbf{inl}(\star) \\ \mathbf{green}^\bullet \gamma \gamma^\bullet &= \mathbf{inr}(\star) \end{aligned}$$

Projection from the computability model There is an evident projection $\pi : \bar{\mathcal{C}} \rightarrow \mathcal{C}$ which merely forgets the realizers; it is easy to see that it is a homomorphism of $\mathbb{T}_{\mathbf{cwf}}^\Pi$ -algebras.

5.2 The canonicity theorem

In this section, we let \mathcal{C} be the *initial* model for $\mathbb{T}_{\mathbf{cwf}}^\Pi$.

Theorem 5.3 (Canonicity). *If $\cdot \vdash_{\mathbb{T}_{\mathbf{cwf}}^\Pi} M : \mathbf{El}(\cdot \vdash \mathbf{color})$, then either $M = \mathbf{red}$ or $M = \mathbf{green}$.*

Proof. Because $\bar{\mathcal{C}}$ is a model of $\mathbb{T}_{\mathbf{cwf}}^\Pi$, we have some (N, N^\bullet) in the interpretation of M such that $\pi M = N$ and $N^\bullet(\cdot)(\star)$ is an element of the following set:

$$\{\star \mid N = \mathbf{red}\} + \{\star \mid N = \mathbf{green}\}$$

Therefore, it suffices to observe that $M = N$; this is guaranteed by the universal property of the initial $\mathbb{T}_{\mathbf{cwf}}^\Pi$ -algebra \mathcal{C} and the fact that π is a homomorphism of $\mathbb{T}_{\mathbf{cwf}}^\Pi$ -algebras.

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